Chapter 5 Conservation of Mass: Control Volume Approach
Main Topics

✓ Introduction
✓ Derivation of Continuity Equation
✓ Applications
5.0 Introduction

In this chapter and the two follow, the fundamental laws mentioned in the previous chapter will be derived without the characteristics of fluid, i.e. the detailed picture of what happens inside the control volume (the behaviour of fluid particle) is of no concern to our present scope.

Fully understanding of Control volume and its control surface will be essential.
5.1 Derivation of Continuity Equation

The law of conservation of mass states that mass may be neither created nor destroyed. With respect to a control volume, the law of conservation of mass may be simply stated as

\[
\left\{ \text{Rate of mass efflux from control volume} \right\} - \left\{ \text{Rate of mass flow into control volume} \right\} + \left\{ \text{Rate of mass accumulated within C.V.} \right\} = 0
\]
5.1 Derivation of Continuity Equation

Fig.5.1.1

Rate of mass efflux = (ρv)(dAcosθ)

(ρv)(dAcosθ) = ρ (dA) |v| |n| cosθ

(v·n) = |v||n|cos θ

Rate of mass efflux = ∫∫_{c.s.} ρ (v·n) dA
Fig. 5.1.1
5.1 Derivation of Continuity Equation

Rate of accumulation of mass = \( \frac{\partial}{\partial t} \iiint_{c.v.} \rho \, dV \)

Integral expression of mass balance over a control volume becomes

\[ \iiint_{c.s.} \rho (\vec{v} \cdot \vec{n}) dA + \frac{\partial}{\partial t} \iiint_{c.v.} \rho dV = 0 \]
5.1 Derivation of Continuity Equation

If flow is steady relative to coordinates fixed to the CS, then
\[ \iint_{c.s.} \rho (\vec{v} \cdot \vec{n}) \, dA = 0 \]

For incompressible flow, the density is constant, then
\[ \iint_{c.s.} (\vec{v} \cdot \vec{n}) \, dA = 0 \]
5.1 Derivation of Continuity Equation

Derivation of (eq.5.1.1) from (eq.4.6.5)

N is M, \( \eta \) is unity (1),

\[
\frac{DM}{Dt} = 0 = \iint_{c.s.} \rho (\vec{v} \cdot \vec{n}) dA + \frac{\partial}{\partial t} \iiint_{c.v.} \rho dV
\]
5.2 Applications

Example 5.2.1
Fig.5.2.1

Example 5.2.2
Fig.5.2.2
Example 5.2.1. Consider the very common situation in which fluid enters some device through a pipe and leaves the device through a second pipe as shown diagrammatically in Fig.5.2.1. The control surface we have chosen is indicated by a dotted line. We shall assume that the flow is steady relative to the control volume and that the inlet and outlet flows are one dimensional. Applying (eq.5.1.2) for the case, we get

\[ \int \int_{A_1} \rho (\vec{v} \cdot \hat{n}) \, dA = \int \int_{A_1} \rho (\vec{v} \cdot \hat{n}) \, dA + \int \int_{A_2} \rho (\vec{v} \cdot \hat{n}) \, dA = 0 \]

Where \( A_1 \) and \( A_2 \) are respectively the entrance and exit areas. Upon noting that the velocities are normal to the control surfaces at these areas. We have

\[ \int \int_{A_1} \rho (\vec{v} \cdot \hat{n}) \, dA = - \int \int_{A_1} \rho \, v \, dA + \int \int_{A_2} \rho \, v \, dA = 0 \]

With \( \rho \) and \( v \) constant at each section as a result of one dimensional restriction for the inlet and outlet flows, we get

\[ - \rho_1 v_1 \int \int_{A_1} dA + \rho_2 v_2 \int \int_{A_2} dA = 0 \]

Integrating, we get

\[ \rho_1 v_1 A_1 = \rho_2 v_2 A_2 \]  
(eq.5.2.1)
Control Volume

Fig. 5.2.1
Example 5.2.2. Let us consider the case of an incompressible flow, for which the flow area is circular and the velocity profile is parabolic, varying according to the expression,

$$v = v_{\text{max}} \left[1 - \left(\frac{r}{R}\right)^2\right]$$

where $v_{\text{max}}$ is the maximum velocity which exists at the centre of the circular passage (i.e. $r = 0$), and $R$ is the radial distance to the inside surface of the circular area considered. The above velocity profile expression represents the velocity at a radial distance $r$ from the centre of the flow section. Since the average velocity is of particular interest in engineering problems, we will now consider the means of obtaining the average velocity from the expression. At the station where this velocity profile exists, the mass rate of flow is

$$(\rho v)_{\text{ave}} A = \int \int_A \rho v \, dA$$

For the present case of incompressible flow the density is constant. Solving for the average velocity we have

$$v_{\text{ave}} = \frac{1}{A} \int \int_A v \, dA$$

$$= \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R v_{\text{max}} \left[1 - \left(\frac{r}{R}\right)^2\right] r \, dr \, d\theta$$

$$= v_{\text{max}} / 2$$
\[ dA = A_y - A_0 \]
\[ = \frac{1}{2} (r^2 + 2(r dr) + (dr)^2) d\theta - \frac{1}{2} r^2 d\theta \]
\[ = \frac{1}{2} (2r (dr) + (dr)^2) d\theta \]
\[ = r (dr) d\theta \]

*Fig. 5.2.3*
Points to remember

✓ The law of conservation of mass can be written mathematically in the following form,

\[
\frac{DM}{Dt} = 0 = \iint_{c.s.} \rho \, (\vec{V} \cdot \vec{n}) \, dA + \frac{\partial}{\partial t} \iiint_{c.v.} \rho \, dV
\]

✓ The first term on the right hand side of the above equation considers the net mass efflux across the control surface of the control volume whereas the second term considers the time rate of change of mass within the control volume.
Tutorial

Link to Tutorial 3
1. In the incompressible flow through the device shown (in Fig.T.3.1), velocities may be considered uniform over the inlet and outlet sections. If the fluid flowing is water, determine an expression for the mass flow rate at section 3. The following conditions are known: $A_1 = 0.1 \text{ m}^2$, $A_2 = 0.2 \text{ m}^2$, $A_3 = 0.15 \text{ m}^2$, $v_1 = 5 \text{ m/sec}$, and $v_2 = 10 + 5 \cos (4\pi t) \text{ m/sec}$.

2. Water enters a wide, flat channel of width, $2h$, with a uniform velocity of $5 \text{ m/sec}$. At the outlet of the channel the velocity distribution is given by

$$\frac{u}{u_{\text{max}}} = 1 - \left[ \frac{y}{h} \right]^2$$

The coordinate $y$ is measured from the centerline of the channel. Determine the exit centerline velocity $u_{\text{max}}$.

3. A two dimensional reducing bend has a linear velocity profile at section 1 in Fig.T.3.3. The flow is uniform at sections 2 and 3. The fluid is incompressible and the flow is steady. Find the magnitude and direction of the uniform velocity at section 3.

4. Water enters a two dimensional channel of constant width, $h$, with a uniform velocity, $U$. The channel makes a $90^\circ$ bend that distorts the flow to produce the velocity profile shown in Fig.T.3.4 at the exit. Evaluate the constant, $C$.

5. A section of pipe carrying water contains an expansion chamber with a free surface whose area is $2 \text{ m}^2$ as shown in Fig.T.3.5. The inlet and outlet pipes are both $1 \text{ m}^2$ in area. At a given instant, the velocity at section 1 is $3 \text{ m/sec}$ into the chamber. Water flows out at section 2 at $4 \text{ m}^3/\text{sec}$. Both flows are uniform. Find the rate of change of free surface level at the given instant. Indicate whether the level rises or falls.

6. A cylindrical tank, $D = 50 \text{ mm}$, drains through an opening, $d = 5 \text{ mm}$, in the bottom of the tank. The speed of the liquid leaving the tank may be approximated by $v = \sqrt{2gy}$, where $y$ is the height from the tank bottom to the free surface. If the tank is initially filled with water to a depth $y_0 = 0.4 \text{ m}$, determine the water depth at time $t = 12 \text{ sec}$.
Tutorial 3

**Fig. T.3.1**

**Fig. T.3.3**

**Fig. T.3.4**

**Fig. T.3.5**