• Tutorial #2
• WRF#16.2; WWWR#17.13, WRF#16.1; WRF#17.39.
• To be discussed on Jan. 29, 2019.
• By either volunteer or class list.
• Tutorial #2
• WWWR
• #17.39: Line 2: The fins are made of aluminum, they are **0.3cm** thick each.
Steady-State Conduction
One-Dimensional Conduction

Steady-state conduction, no internal generation of energy

\[ \nabla^2 T = 0 \]

For one-dimensional, steady-state transfer by conduction

i = 0 rectangular coordinates
i = 1 cylindrical coordinates
i = 2 spherical coordinates

\[ \frac{d}{dx} \left( x^i \frac{dT}{dx} \right) = 0 \]
Figure 17.1 Plane wall with a one-dimensional temperature distribution.

**Plane Wall.** Consider the conduction of energy through a plane wall as illustrated in Figure 17.1. The one-dimensional Laplace equation is easily solved, yielding

\[ T = C_1x + C_2 \quad (17-2) \]

The two constants are obtained by applying the boundary conditions

at \( x = 0 \) \quad \( T = T_1 \)

and

at \( x = L \) \quad \( T = T_2 \)

These constants are

\[ C_2 = T_1 \]

and

\[ C_1 = \frac{T_2 - T_1}{L} \]
Composite Walls. The steady flow of energy through several walls in series is often encountered. A typical furnace design might include one wall for strength, an intermediate wall for insulation, and the third outer wall for appearance. This composite plane wall is illustrated in Figure 17.2.

For a solution to the system shown in this figure, the reader is referred to section 15.5.

The following example illustrates the use of the composite-wall energy-rate equation for predicting the temperature distribution in walls.

Figure 17.2 Temperature distribution for steady-state conduction of energy through a composite plane wall.
A furnace wall is composed of three layers, 10 cm of firebrick \((k = 1.560 \text{ W/m } \cdot \text{ K})\), followed by 23 cm of kaolin insulating brick \((k = 0.073 \text{ W/m } \cdot \text{ K})\), and finally 5 cm of masonry brick \((k = 1.0 \text{ W/m } \cdot \text{ K})\). The temperature of the inner wall surface is 1370 K and the outer surface is at 360 K. What are the temperatures at the contacting surfaces?

The individual material thermal resistances per \(m^2\) of area are

\[
R_1, \ \text{firebrick} = \frac{L_1}{k_1A_1} = \frac{0.10 \text{ m}}{(1.560 \text{ W/m } \cdot \text{ K})(1 \text{ m}^2)} = 0.0641 \text{ K/W}
\]

\[
R_2, \ \text{kaolin} = \frac{L_2}{k_2A_2} = \frac{0.23}{(0.0173)(1)} = 13.3 \text{ K/W}
\]

\[
R_3, \ \text{masonry} = \frac{L_3}{k_3A_3} = \frac{0.05}{(1.0)(1)} = 0.05 \text{ K/W}
\]

The total resistance of the composite wall is equal to \(0.0641 + 13.3 + 0.05 = 13.41 \text{ K/W}\).

The total temperature drop is equal to \((T_1 - T_2) = 1370 - 360 = 1010 \text{ K}\).

Using equation (15-16), the energy transfer rate is

\[
q = \frac{T_1 - T_4}{\sum R} = \frac{1010 \text{ K}}{13.41 \text{ K/W}} = 75.3 \text{ W}
\]

Since this is a steady-state situation, the energy transfer rate is the same for each part of the transfer path (i.e. through each wall section). The temperature at the firebrick-kaolin interface, \(T_2\), is given by

\[
T_1 - T_2 = q(R_1) = (75.3 \text{ W})(0.0641 \text{ K/W}) = 4.83 \text{ K}
\]

giving

\[
T_2 = 1365.2 \text{ K}
\]
Similarly,

\[
T_3 - T_4 = q(R_3)
\]

\[
= (75.3 \text{ W})(0.05 \text{ K/W}) = 3.77 \text{ K}
\]

giving

\[
T_3 = 363.8 \text{ K}
\]

There are numerous situations in which a composite wall involves a combination of series and parallel energy-flow paths. An example of such a wall is illustrated in Figure 17.3, where steel is used as reinforcement for a concrete wall. The composite wall can be divided into three sections of length \(L_1\), \(L_2\), and \(L_3\), and the thermal resistance for each of these lengths may be evaluated.
Equivalent resistance of the parallel resistors $R_a$ and $R_b$ is $R_c$
The intermediate layer between planes 2 and 3 consists of two separate thermal paths in parallel; the effective thermal conductance is the sum of the conductances for the two materials. For the section of the wall of height $y_1 + y_2$ and unit depth, the resistance is

$$R_2 = \frac{1}{\frac{k_1y_1}{L_2} + \frac{k_2y_2}{L_2}} = L_2 \left( \frac{1}{\frac{k_1y_1}{L_2} + \frac{k_2y_2}{L_2}} \right)$$

The total resistance for this wall is

$$\sum R_T = R_1 + R_2 + R_3$$

or

$$\sum R_T = \frac{L_1}{k_1(y_1 + y_2)} + L_2 \left( \frac{1}{\frac{k_1y_1}{L_2} + \frac{k_2y_2}{L_2}} \right) + \frac{L_3}{k_1(y_1 + y_2)}$$

The electrical circuit is an analog to the composite wall.

The rate of energy transferred from plane 1 to plane 4 is obtained by a modified form of equation (15-16).

$$q = \frac{T_1 - T_4}{\sum R_T} = \frac{T_1 - T_4}{\frac{L_1}{k_1(y_1 + y_2)} + L_2 \left( \frac{1}{\frac{k_1y_1}{L_2} + \frac{k_2y_2}{L_2}} \right) + \frac{L_3}{k_1(y_1 + y_2)}}$$

(17-5)
**Long, Hollow Cylinder.** Radial energy flow by conduction through a long, hollow cylinder is another example of one-dimensional conduction. The radial heat flow for this configuration was evaluated in example 15.1 as

\[
\frac{q_r}{L} = \frac{2\pi k}{\ln(r_o/r_i)} (T_1 - T_0)
\]  

(17-6)

where \( r_i \) is the inside radius, \( r_o \) is the outside radius, \( T_i \) is the temperature on the inside surface, and \( T_o \) is the temperature on the outside surface. The resistance concept may again be used; the thermal resistance of the hollow cylinder is

\[
R = \frac{\ln(r_o/r_i)}{2\pi k L}
\]  

(17-7)

The radial temperature distribution in a long, hollow cylinder may be evaluated by using equation (17-1) in cylindrical form

\[
\frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0
\]  

(17-8)
Solving this equation subject to the boundary conditions

\[ \text{at } r = r_i \quad T = T_i \]

and

\[ \text{at } r = r_o \quad T = T_0 \]

we see that temperature profile is

\[ T(r) = T_i - \frac{T_i - T_o}{\ln(r_o/r_i)} \ln \frac{r}{r_i} \quad (17-9) \]

Thus the temperature in a long, hollow cylinder is a logarithmic function of radius \( r \), while for the plane wall the temperature distribution is linear.

The following example illustrates the analysis of radial energy conduction through a long, hollow cylinder.
A long steam pipe of outside radius $r_2$ is covered with thermal insulation having an outside radius of $r_3$. The temperature of the outer surface of the pipe, $T_2$, and the temperature of the surrounding air, $T_\infty$, are fixed. The energy loss per unit area of outside surface of the insulation is described by the Newton rate equation

$$\frac{q_r}{A} = h(T_3 - T_\infty)$$  \hspace{1cm} (15-11)

Figure 17.4 A series composite hollow cylinder.
Can the energy loss increase with an increase in the thickness of insulation? If possible, under what conditions will this situation arise? Figure 17.4 may be used to illustrate this composite cylinder.

In example 15.3, the thermal resistance of a hollow cylindrical element was shown to be

\[ R = \frac{\ln(r_o/r_i)}{2\pi kL} \]  

(17-10)

In the present example, the total difference in temperature is \( T_2 - T_x \) and the two resistances, due to the insulation and the surrounding air film, are

\[ R_2 = \frac{\ln(r_3/r_2)}{2\pi k_2L} \]

for the insulation, and

\[ R_3 = \frac{1}{hA} = \frac{1}{h2\pi r_3L} \]

for the air film.

In the case of 85% magnesia insulation \((k = 0.0692 \text{ W/m} \cdot \text{K})\) and a typical value for the heat transfer coefficient in natural convection \((h = 34 \text{ W/m}^2 \cdot \text{K})\), the critical radius is calculated as

\[ r_{\text{crit}} = \frac{k}{h} = \frac{0.0692 \text{ W/m} \cdot \text{K}}{34 \text{ W/m}^2 \cdot \text{K}} = 0.0020 \text{ m} \quad (0.0067 \text{ ft}) \]

\[ = 0.20 \text{ cm} \quad (0.0787 \text{ in.}) \]
Substituting these terms into the radial heat flow equation and rearranging, we obtain

\[ q_r = \frac{2\pi L(T_2 - T_\infty)}{[\ln(r_3/r_2)]/k_2 + 1/hr_3} \quad (17-11) \]

The dual effect of increasing the resistance to energy transfer by conduction and simultaneously increasing the surface area as \( r_3 \) is increased suggests that, for a pipe of given size, a particular outer radius exists for which the heat loss is maximum. Since the ratio \( r_3/r_2 \) increases logarithmically, and the term \( 1/r_3 \) decreases as \( r_3 \) increases, the relative importance of each resistance term will change as the insulation thickness is varied. In this example, \( L, T_2, T_\infty, k_2, h, \) and \( r_2 \) are considered constant. Differentiating equation (17-11) with respect to \( r_3 \), we obtain

\[ \frac{dq_r}{dr_3} = \frac{2\pi L(T_2 - T_\infty)\left(\frac{1}{k_2 r_3} - \frac{1}{hr_3^2}\right)}{\left[\frac{1}{k_2} \ln \left(\frac{r_3}{r_2}\right) + \frac{1}{hr_3}\right]^2} \quad (17-12) \]

The radius of insulation associated with the maximum energy transfer, the critical radius, found by setting \( dq_r/dr_3 = 0 \); equation (17-12) reduces to

\[ (r_2)_{\text{critical}} = \frac{k_2}{h} \quad (17-13) \]
Thus, insulating the pipe may actually increase the rate of heat transfer instead of decreasing it.

\[ r_{cr,\text{cylinder}} = \frac{k}{h} \quad (\text{m}) \]
\[ \varepsilon_r = \frac{2\varepsilon L (z_L - z_0)}{\varepsilon L (\frac{1}{k_3} + \frac{1}{h r_3})} \rightarrow f (>0) \text{ constant} \]

Max \( \varepsilon_r \) occurs when \( f \) is at the min

\[ f = \left[ \frac{\varepsilon_0}{k_3} \right] / k_3 + \frac{1}{h r_3} \]

\[ \frac{df}{dy_3} = \left( - \frac{1}{y_3} \right) \frac{1}{k_3} + \frac{(-1)}{h r_3^2} \]

\[ = \frac{-1}{y_3 k_3} - \frac{1}{h r_3^2} \]

\[ \frac{df}{dy_3} \bigg|_{y_3 \text{ critical}} = 0 \quad \Rightarrow \quad y_3 = \frac{k_2}{h} \]

\[ \text{Eq. (17-13)} \]

\[ \frac{\alpha^2 f}{dy_3^2} = \frac{-1}{y_3^2 k_3^2} + \frac{2}{h r_3^3} \]

\[ \left. \frac{d^2 f}{dy_3^2} \right|_{y_3 = y_3 \text{ critical}} = \frac{-1}{k_2 - \frac{k_2^2}{h^2}} + \frac{2}{h^2 \frac{k_2}{h}} \]

\[ = \frac{-h^2}{k_2^3} + \frac{2h}{k_2^3} \]

\[ = \frac{h^2}{k_2^3} > 0 \]

\[ \Rightarrow \quad f \text{ is min at } y_3 = y_3 \text{ critical} \]

\[ \Rightarrow \quad \varepsilon_r \text{ is max at } y_3 = y_3 \text{ critical} \]
Hollow Sphere. Radial heat flow through a hollow sphere is another example of one-dimensional conduction. For constant thermal conductivity, the modified Fourier rate equation

\[ q_r = -k \frac{dT}{dr} A \]

applies, where \( A = \text{area of a sphere} = 4 \pi r^2 \), giving

\[ q_r = -4\pi k r^2 \frac{dT}{dr} \quad (17-14) \]

This relation, when integrated between the boundary conditions

at \( T = T_i \quad r = r_i \)

and

at \( T = T_o \quad r = r_o \)

yields

\[ q = \frac{4\pi k (T_i - T_o)}{\frac{1}{r_i} - \frac{1}{r_o}} \quad (17-15) \]

The hyperbolic temperature distribution

\[ T = T_i - \left( \frac{T_i - T_o}{1/r_i - 1/r_o} \right) \left( \frac{1}{r_i} - \frac{1}{r} \right) \quad (17-16) \]
For steady-state conduction in the x direction without internal generation of energy, the equation which applies is

$$\frac{d}{dx} \left( k \frac{dT}{dx} \right) = 0$$

Where $k$ may be a function of $T$.

In many cases the thermal conductivity may be a linear function of temperature over a considerable range. The equation of such a straight-line function may be expressed by

$$k = k_o (1 + \beta T)$$

Where $k_o$ and $\beta$ are constants for a particular material.
Cylindrical Solid with Homogeneous Energy Generation. Consider a cylindrical solid with internal energy generation as shown in Figure 17.5. The cylinder will be considered long enough so that only radial conduction occurs. The density, $\rho$, the heat capacity, $c_p$, and the thermal conductivity of the material will be considered constant. The energy balance for the element shown is

$$\begin{align*}
\text{Rate of energy conduction into the element} &+ \text{rate of energy generation within the element} \\
&- \text{rate of energy conduction out of the element}
\end{align*}$$

(17-18)

$$\begin{align*}
&= \text{rate of accumulation of energy} \\
&\text{within the element}
\end{align*}$$

Figure 17.5 Annular element in a long, circular cylinder with internal heat generation.
Applying the Fourier rate equation and letting \( \dot{q} \) represent the rate of energy generated per unit volume, we may express equation (17-18) by the algebraic expression

\[
- k (2 \pi r L) \frac{\partial T}{\partial r} \bigg|_r + \dot{q} (2 \pi r L \Delta r) - \left[ - k (2 \pi r L) \frac{\partial T}{\partial r} \right]_{r+\Delta r} = \rho c_p \frac{\partial T}{\partial t} (2 \pi r L \Delta r)
\]

Dividing each term by \( 2 \pi r L \Delta r \), we obtain

\[
\dot{q} + \frac{k [r(\partial T / \partial r)|_{r+\Delta r} - r(\partial T / \partial r)|_r]}{r \Delta r} = \rho c_p \frac{\partial T}{\partial t}
\]

In the limit as \( \Delta r \) approaches zero, the following differential equation is generated:

\[
\dot{q} + \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \rho c_p \frac{\partial T}{\partial t} \quad (17-19)
\]

For steady-state conditions, the accumulation term is zero; when we eliminate this term from the above expression, the differential equation for a solid cylinder with homogeneous energy generation becomes

\[
\dot{q} + \frac{k}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0 \quad (17-20)
\]

The variables in this equation may be separated and integrated to yield

\[
 rk \frac{dT}{dr} + \frac{\dot{q} r^2}{2} = C_1
\]

or

\[
k \frac{dT}{dr} + \frac{\dot{q} r}{2} = \frac{C_1}{r}
\]
Because of the symmetry of the solid cylinder, a boundary condition which must be satisfied stipulates that the temperature gradient must be finite at the center of the cylinder, where \( r = 0 \). This can only be true if \( C_1 = 0 \). Accordingly, the above relation reduces to

\[
k \frac{dT}{dr} + \frac{\dot{q}}{2} r = 0 \tag{17-21}
\]

A second integration will now yield

\[
T = -\frac{\dot{q} r^2}{4k} + C_2 \tag{17-22}
\]

If the temperature \( T \) is known at any radial value, such as a surface, the second constant, \( C_2 \), may be evaluated. This, of course, provides the completed expression for the temperature profile. The energy flux in the radial direction may be obtained from

\[
\frac{q_r}{A} = -k \frac{dT}{dr}
\]

by substituting equation (17-21), yielding

\[
\frac{q_r}{A} = \frac{\dot{q} r}{2}
\]

or

\[
q_r = (2\pi r L) \frac{\dot{q} r}{2} = \pi r^2 L \dot{q} \tag{17-23}
\]
Plane Wall with Variable Energy Generation

\[ q = q_L \left[ 1 + \beta (T - T_L) \right] \]

The symmetry of the temperature distribution requires a zero temperature gradient at \( x = 0 \).

The case of steady-state conduction in the \( x \) direction in a stationary solid with constant thermal conductivity becomes

\[
\frac{d^2T}{dx^2} + \frac{\dot{q}_L}{k} \left[ 1 + \beta (T - T_L) \right] = 0
\]
Figure 17.6 Flat plate with temperature-dependent energy generation.

The boundary conditions are

$$\text{at } x = 0 \quad \frac{dT}{dx} = 0$$
and

\[ x = \pm L \quad T = T_L \]

These relations may be expressed in terms of a new variable, \( \theta = T - T_L \), by

\[ \frac{d^2 \theta}{dx^2} + \frac{\dot{q}_L}{k} (1 + \beta \theta) = 0 \]

or

\[ \frac{d^2 \theta}{dx^2} + C + s \theta = 0 \]

where \( C = \dot{q}_L/k \) and \( s = \beta \dot{q}_L/k \). The boundary conditions are

at \( x = 0 \) \quad \frac{d\theta}{dx} = 0

and

at \( x = \pm L \) \quad \theta = 0

The integration of this differential equation is simplified by a second change in variables; inserting \( \phi \) for \( C + s \theta \) into the differential equation and the boundary conditions; we obtain

\[ \frac{d^2 \phi}{dx^2} + s \phi = 0 \]
Detailed derivation for the transformation

\[ \Phi = C + s \theta \]
for

\[ x = 0 \quad \frac{d\phi}{dx} = 0 \]

and

\[ x = \pm L \quad \phi = C \]

The solution is

\[ \phi = C + s\theta = A \cos(x\sqrt{s}) + B \sin(x\sqrt{s}) \]

or

\[ \theta = A_1 \cos(x\sqrt{s}) + A_2 \sin(x\sqrt{s}) - \frac{C}{s} \]

The temperature distribution becomes

\[ T - T_L = \frac{1}{\beta} \left[ \frac{\cos(x\sqrt{s})}{\cos(L\sqrt{s})} - 1 \right] \]

(17-25)

where \( s = \beta q_L/k \) is obtained by applying the two boundary conditions.

The cylindrical and spherical examples of one-dimensional temperature-dependent generation are more complex; solutions to these may be found in the technical literature.
Plane Wall with Variable Energy Generation
(to show eqn 17.25)

Given: \( \dot{q} = \dot{q}_L \left[ 1 + \beta (T - T_L) \right] \)  \hspace{1cm} (17.24)
For steady state, constant \( k \)
\( \nabla^2 T + \frac{1}{k} \dot{q} = 0 \) \hspace{1cm} (Poisson eqn)

Boundary conditions: \( x = 0, \frac{dT}{dx} = 0 \)
\( x = \pm L, \) , \( T = T_L \)

change in variable:
\( \Theta = T - T_L \)
\( \frac{d^2 \Theta}{dx^2} + \frac{\dot{q}_L}{k} (1 + \beta \Theta) = 0 \)
\( \frac{d^2 \Theta}{dx^2} + C + s \Theta = 0 \)
where \( C = \frac{\dot{q}_L}{k} \), \( s = \frac{\beta \dot{q}_L}{k} \)
\( \text{BC:} \hspace{1cm} x = 0, \frac{d\Theta}{dx} = 0 \) \hspace{1cm} \( x = \pm L, \) \( \Theta = 0 \)

and change in variable: \( \text{let} \) \( \phi = C + s \Theta \)
\( \frac{d^2 \phi}{dx^2} + s \phi = 0 \)
\( \text{BC:} \hspace{1cm} x = 0, \frac{d\phi}{dx} = 0 \)
\( x = \pm L; \phi = C \)
\( \frac{d^2 \phi}{dx^2} + 0 \frac{d\phi}{dx} + s \phi = 0 \)

Auxiliary Equation: \( m^2 + 0m + s = 0 \)
\( m = \pm \sqrt{-s} = \pm i \sqrt{-s} \)

\( \text{General Solution:} \)
\( \phi = C + s \Theta = A \sin(Te x) + B \cos(Te x) \)
Detailed Derivation for Equations 17-25

Courtesy by all CN5 Grace Mok, 2003-2004
Heat Transfer from Finned Surfaces

- Temperature gradient $dT/dx$,
- Surface temperature, $T$,
- Are expressed such that $T$ is a function of $x$ only.
- Newton’s law of cooling

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

- Two ways to increase the rate of heat transfer:
  - increasing the heat transfer coefficient,
  - increase the surface area $\rightarrow$ fins
- Fins are the topic of this section.

The shaded area represents a portion of the extended surface which has variable cross-sectional area, \( A(x) \), and surface area, \( S(x) \), which are functions of \( x \) alone. For steady-state conditions the first law of thermodynamics, equation (6-10), reduces to the simple expression

\[
\frac{\delta Q}{dt} = 0
\]

thus, in terms of the heat flow rates designated in the figure, we may write

\[
q_1 = q_2 + q_3
\]
The quantities $q_1$ and $q_2$ are conduction terms, while $q_3$ is a convective heat-flow rate. Evaluating each of these in the appropriate way and substituting into equation (17-26), we obtain

$$kA \frac{dT}{dx} \bigg|_{x=\Delta x} - kA \frac{dT}{dx} \bigg|_{x} - hS(T - T_\infty) = 0 \quad (17-27)$$

where $T_\infty$ is the fluid temperature. Expressing the surface area, $S(x)$, in terms of the width, $\Delta x$, times the perimeter, $P(x)$, and dividing through by $\Delta x$, we obtain

$$\frac{kA(dT/dx)|_{x+\Delta x} - kA(dT/dx)|_{x}}{\Delta x} - hP(T - T_\infty) = 0$$

Evaluating this equation in the limit as $\Delta x \to 0$, we obtain the very general differential equation

$$\frac{d}{dx} \left( kA \frac{dT}{dx} \right) - hP(T - T_\infty) = 0 \quad (17-28)$$
For constant cross section and constant thermal conductivity

\[
\frac{d^2 \theta}{dx^2} - m^2 \theta = 0 \quad \text{(A)}
\]

Where

\[
\theta = T - T_\infty \quad ; \quad m^2 = \frac{hp}{kA_c} ; \quad A = A_c
\]

- Equation (A) is a linear, homogeneous, second-order differential equation with constant coefficients.
- The general solution of Eq. (A) is

\[
\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \quad \text{(B)}
\]

- \(C_1\) and \(C_2\) are constants whose values are to be determined from the boundary conditions at the base and at the tip of the fin.
\[ \sinh A = (e^A - e^{-A})/2 \]
\[ \cosh A = (e^A + e^{-A})/2 \]
\[ \tanh A = \frac{\sinh(A)}{\cosh(A)} \]
Boundary Conditions

Several boundary conditions are typically employed:

- **At the fin base**
  - **Specified temperature** boundary condition, expressed as: \( \theta(0) = \theta_b = T_b - T_\infty \)

- **At the fin tip**
  1. Specified temperature
  2. Infinitely Long Fin
  3. Adiabatic tip
  4. Convection (and combined convection)

(1) Fins or Spines of Uniform Cross Section. For either of the cases shown in Figure 17.8 the following are true: \( A(x) = A \), and \( P(x) = P \), both constants. If, additionally, both \( k \) and \( h \) are taken to be constant, equation (17-28) reduces to

\[
\frac{d^2T}{dx^2} - \frac{hP}{kA} (T - T_\infty) = 0
\]

(17-29)

Figure 17.8 Two examples of extended surfaces with constant cross section.
(2) *Straight Surfaces with Linearly Varying Cross Section.* Two configurations for which $A$ and $P$ are not constant are shown in Figure 17.9. If the area and perimeter both vary in a linear manner from the primary surface, $x = 0$, to some lesser value at the end, $x = L$, both $A$ and $P$ may be expressed as

\[
A = A_0 - (A_0 - A_L) \frac{x}{L} \tag{17-30}
\]

and

\[
P = P_0 - (P_0 - P_L) \frac{x}{L} \tag{17-31}
\]

In the case of the rectangular fin shown in Figure 17.9(b), the appropriate values of $A$ and $P$ are

\[
A_0 = 2t_0W \quad A_L = 2t_LW \\
P_0 = 2[2t_0 + W] \quad P_L = 2[2t_L + W]
\]

where $t_0$ and $t_L$ represent the semithickness of the fin evaluated at $x = 0$ and $x = L$, respectively, and $W$ is the total depth of the fin.
Figure 17.9 Two examples of straight extended surfaces with variable cross section.

For constant $h$ and $k$, equation (17-28) applied to extended surfaces with cross-sectional area varying linearly becomes

$$\left[ A_0 - (A_0 - A_L) \frac{x}{L} \right] \frac{d^2 T}{dx^2} - \frac{A_0 - A_L}{L} \frac{dT}{dx} - \frac{h}{k} \left[ P_0 - (P_0 - P_L) \frac{x}{L} \right] (T - T_\infty) = 0 \quad (17-32)$$
How to derive the functional dependence of $A_c = A = A(x)$ for a straight fin with variable cross section area?
General Solution for Straight Fin with Three Different Boundary Conditions

\[ \theta = A \cosh mx + B \sinh mx \quad (17-35) \]

where \( m^2 = \frac{hP}{kA} \) and \( \theta = T - T_\infty \). The evaluation of the constants of integration requires that two boundary conditions be known. The three sets of boundary conditions which we shall consider are as follows:

(a) \( T = T_0 \) \quad \text{at} \quad x = 0

\( T = T_L \) \quad \text{at} \quad x = L

(b) \( T = T_0 \) \quad \text{at} \quad x = 0

\( \frac{dT}{dx} = 0 \) \quad \text{at} \quad x = L

(c) \( T = T_0 \) \quad \text{at} \quad x = 0

\(-k \frac{dT}{dx} = h(T - T_\infty) \) \quad \text{at} \quad x = L
In set (a)
   Known temperature at $x = L$
In set (b)
   Temperature gradient is zero at $x = L$
In set (c)
   Heat flow to the end of an extended surface by conduction be equal to that leaving this position by convection.
The temperature profile, associated with the first set of boundary conditions, is

\[
\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \left(\frac{\theta_L}{\theta_0} - e^{-ml}\right)\left(\frac{e^{mx} - e^{-mx}}{e^{mx} - e^{-mx}}\right) + e^{-mx}
\]  

(17-36)

A special case of this solution applies when \( L \) becomes very large, that is, \( L \to \infty \), for which equation (17-36) reduces to

\[
\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = e^{-mx}
\]  

(17-37)

The constants, \( c_1 \) and \( c_2 \), obtained by applying set (b), yield, for the temperature profile

\[
\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{e^{mx}}{1 + e^{2ml}} + \frac{e^{-mx}}{1 + e^{-2ml}}
\]  

(17-38)

An equivalent expression to equation (17-38) but in more compact form is

\[
\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh[m(L - x)]}{1 + e^{2ml}}
\]  

(17-39)

Note that, in either equation (17-38) or (17-39), as \( L \to \infty \) the temperature profile approaches that expressed in equation (17-37).

The application of set (c) of the boundary conditions yields, for the temperature profile

\[
\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh[m(L - x)] + (h/mk)\sinh[m(L - x)]}{\cosh mL + (h/mk)\sinh mL}
\]  

(17-40)
Detailed Derivation for Equations 17-36 (Case a).

Courtesy by CN3 Yeong Sai Hooi 2002-2003
\[ \theta = c_1 e^{mx} + c_2 e^{-mx}, \quad \theta = T - T_0, \quad m^2 = \frac{k_p}{\varepsilon A} \]

For \( T = T_0 \) or \( x = 0 \),
\[ \theta_0 = T_0 - T_0 = c_1 e^{m(0)} + c_2 e^{-m(0)} \]
\[ \theta_0 = c_1 + c_2 \]
\[ \theta_0 = \theta_0 - c_2 \]

Since \( \theta = c_1 e^{mx} + c_2 e^{-mx} \)
\[ \frac{d\theta}{dx} = mc_1 e^{mx} - mc_2 e^{-mx} \]
\[ \frac{d(T - T_0)}{dx} = mc_1 e^{mx} - mc_2 e^{-mx} \]

For \( \frac{dT}{dx} = 0 \) at \( x = L \)
\[ \frac{d^2T}{dx^2} = mc_1 e^{mx} - mc_2 e^{-mx} \]
\[ 0 = m(\theta_0 - c_2) e^{mx} - mc_2 e^{-mx} \]
\[ mc_2(1 + e^{2mx}) = m \theta_0 e^{mx} \]
\[ \Rightarrow c_2 = \frac{\theta_0 e^{mx}}{e^{mx} + e^{-mx}} = \frac{\theta_0}{1 + e^{-2mx}} \]
\[ \Rightarrow c_1 = \theta_0 - c_2 \]
\[ \Rightarrow c_1 = \frac{\theta_0 e^{mx} + \theta_0 e^{-mx} - \theta_0 e^{mx}}{e^{mx} + e^{-mx}} = \frac{\theta_0}{1 + e^{-2mx}} \]

Sub \( c_1 \) & \( c_2 \) into \( \theta_x \)
\[ \theta = \frac{\theta_0}{1 + e^{2mx}} e^{mx} + \frac{\theta_0}{1 + e^{-2mx}} e^{-mx} \]
\[ = \theta_0 \frac{e^{mx}(1 + e^{-2mx}) + \theta_0 e^{-mx}(1 + e^{2mx})}{(1 + e^{2mx})(1 + e^{-2mx})} \]

Sub \( c_1 \) & \( c_2 \) into \( \theta_0 \)
\[ \theta_0 = \frac{\theta_0}{1 + e^{2mx}} + \frac{\theta_0}{1 + e^{-2mx}} \]
\[ = \frac{\theta_0(1 + e^{-2mx}) + \theta_0(1 + e^{2mx})}{(1 + e^{2mx})(1 + e^{-2mx})} \]
\[ = \theta_0 \frac{(1 + e^{-2mx}) + \theta_0(1 + e^{2mx})}{(1 + e^{2mx})(1 + e^{-2mx})} \]

\[ \frac{T - T_0}{T_0 - T_0} = \frac{e^{mx}(1 + e^{-2mx}) + e^{-mx}(1 + e^{2mx})}{(1 + e^{3mx}) + (1 + e^{-3mx})} \]
\[ = \frac{e^{mx}(1 + e^{-2mx}) + e^{-mx}(1 + e^{2mx})}{(1 + e^{2mx})(1 + e^{-2mx})} \]
\[ = \frac{e^{mx} + e^{-mx}}{1 + e^{-2mx}} \]

\[ \theta = \frac{T - T_0}{T_0 - T_0} = \frac{\theta_0}{1 + e^{2mx}} + \frac{\theta_0}{1 + e^{-2mx}} \]

- (17-38)
Detailed Derivation for Equations 17-40 (Case c for extended surface heat transfer).

Courtesy by all CN4 students, presented by Loo Huiyun, 2002-2003
It may be noted that this expression reduces to equation (17-39) if \( d\theta/dx = 0 \) at \( x = L \) and to equation (17-37) if \( T = T_\infty \) at \( L = \infty \).

The expressions for \( T(x) \) which have been obtained are particularly useful in evaluating the total heat transfer from an extended surface. This total heat transfer may be determined by either of two approaches. The first is to integrate the convective heat-transfer expression over the surface according to

\[
q = \int_S h[T(x) - T_\infty] \, dS = \int_S h \theta \, dS
\]

(17-41)

The second method involves evaluating the energy conducted into the extended surface at the base as expressed by

\[
q = -kA \left. \frac{dT}{dx} \right|_{x=0}
\]

(17-42)

The latter of these two expressions is easier to evaluate; accordingly we will use this equation in the following development.
Using equation (17-36), we find that the heat transfer rate, when set (a) of the boundary conditions applies, is

\[ q = kA m \theta_0 \left[ 1 - 2 \frac{\theta_i/\theta_0 - e^{-mL}}{e^{mL} - e^{-mL}} \right] \]  

(17-43)

If the length \( L \) is very long, this expression becomes

\[ q = kA m \theta_0 = kA m (T_0 - T_\infty) \]  

(17-44)

Substituting equation (17-39) [obtained by using set (b) of the boundary conditions] into equation (17-42) we obtain

\[ q = kA m \theta_0 \tanh mL \]  

(17-45)

Equation (17-40), utilized in equation (17-42), yields for \( q \) the expression

\[ q = kA m \theta_0 \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \]  

(17-46)
From expression 17-40,

\[ \theta = T - T_0 = \theta_0 \left\{ \frac{\cosh[m(L-x)] + (\frac{h}{mk}) \sinh[m(L-x)]}{\cosh mL + (\frac{h}{mk}) \sinh mL} \right\} \]

\[ \frac{d\theta}{dx} = \frac{dT}{dx} = \frac{\theta_0}{B} \left\{ -m \sinh[m(L-x)] - \frac{h}{mk} \cosh[m(L-x)] \right\} \]

\[ = - \frac{\theta_0 m}{B} \left\{ \sinh[m(L-x)] + \frac{h}{mk} \cosh[m(L-x)] \right\} \]

Given

\[ q = -kA \frac{dT}{dx} \]

\[ = -kA \left( - \frac{\theta_0 m}{B} \right) \left\{ \sinh[m(L-x)] + \frac{h}{mk} \cosh[m(L-x)] \right\} \]

\[ = kA m \theta_0 \left\{ \sinh[m(L-x)] + \frac{h}{mk} \cosh[m(L-x)] \right\} \]

where \( B = \cosh mL + (\frac{h}{mk}) \sinh mL \).

\[ x = 0 \Rightarrow q = kA m \theta_0 \frac{\sinh mL + (\frac{h}{mk}) \cosh mL}{\cosh mL + (\frac{h}{mk}) \sinh mL} \]
• For a sufficiently long fin the temperature at the fin tip approaches the ambient temperature

Boundary condition: \( \theta(L \to \infty) = T(L) - T_\infty = 0 \)

• When \( x \to \infty \) so does \( e^{mx} \to \infty \)

\( C_1 = 0 \)

• @ \( x=0 \): \( e^{mx} = 1 \)

\( C_2 = \theta_b \)

• The temperature distribution:

\[
\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} = e^{-x\sqrt{hp/kA_c}}
\]

• heat transfer from the entire fin

\[
q = -kA_c \frac{dT}{dx} \bigg|_{x=0} = \sqrt{hpkA_c} \left( T_b - T_\infty \right)
\]
• To maximize the heat transfer from a fin the temperature of the fin should be uniform (maximized) at the base value of $T_b$

• In reality, the temperature drops along the fin, and thus the heat transfer from the fin is less

• To account for the effect we define a fin efficiency

$$\eta_{\text{fin}} = \frac{q_{\text{fin}}}{q_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin}}$$

if the entire fin were at base temperature

or

$$q_{\text{fin}} = \eta_{\text{fin}} q_{\text{fin, max}} = \eta_{\text{fin}} hA_{\text{fin}} (T_b - T_\infty)$$
Fin Efficiency


• For constant cross section of very long fins:

\[
\eta_{\text{long, fin}} = \frac{q_{\text{fin}}}{q_{\text{fin, max}}} = \frac{\sqrt{h pkA_c \left( T_b - T_\infty \right)}}{hA_{\text{fin}} \left( T_b - T_\infty \right)} = \frac{1}{L} \sqrt{\frac{kA_c}{hp}} = \frac{1}{mL}
\]

• For constant cross section with adiabatic tip:

\[
\eta_{\text{adiabatic, fin}} = \frac{q_{\text{fin}}}{q_{\text{fin, max}}} = \frac{\sqrt{h pkA_c \left( T_b - T_\infty \right)} \tanh mL}{hA_{\text{fin}} \left( T_b - T_\infty \right)}
\]

\[
A_{\text{fin}} = P*L = \frac{\tanh mL}{mL}
\]
For constant cross section with adiabatic tip:

\[
\frac{\partial q}{\partial x} = \frac{\cosh [m(l-x)]}{\cosh ml} = \frac{T - T_0}{T_0 - T_0}
\]

\[
\left. \frac{dT}{dx} \right|_{x=0} = \left. \frac{dq}{dx} \right|_{x=0} = (T_0 - T_0) \left. \frac{(-mx) \sinh (m(l-x))}{\cosh ml} \right|_{x=0}
\]

\[
= -m (T_0 - T_0) \frac{\sinh ml}{\cosh ml}
\]

\[
= -m (T_0 - T_0) \tanh (ml)
\]

\[
q = -kAC \left. \frac{dT}{dx} \right|_{x=0} = kAC m (T_0 - T_0) \tanh (ml)
\]

\[
= \sqrt{hP \frac{kAC}{(T_0 - T_0) \tanh (ml)}}
\]

where \( m = \sqrt{hP / kAC} \)

\[
\text{Adiabatic fin: } \frac{q_{\text{fin}}}{q_{\text{fin, max}}} = \frac{\sqrt{hP kAC (T_0 - T_0) \tanh (ml)}}{ha_{\text{fin}} (T_0 - T_0)}
\]

\[
A_{\text{fin}} = P \cdot L
\]

\[
\text{Adiabatic fin: } \frac{T_{\text{fin}}}{T_{\text{fin, max}}} = \frac{T_{\text{fin}}}{T_{\text{fin, max}}} = \frac{\tanh (ml)}{\sqrt{hP / kAC} \cdot L} = \frac{\tanh ml}{mL}
\]
The performance of the fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case.

The performance of fins is expressed in terms of the *fin effectiveness* $\varepsilon_{\text{fin}}$ defined as

$$\varepsilon_{\text{fin}} = \frac{q_{\text{fin}}}{q_{\text{no fin}}} = \frac{q_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{q_{\text{fin}}}{q_{\text{no fin}}}$$

- $q_{\text{fin}}$: Heat transfer rate from the fin of *base area* $A_b$
- $q_{\text{no fin}}$: Heat transfer rate from the surface of *area* $A_b$
(3) Curved Surfaces of Uniform Thickness. A common type of extended surface is that of the circular fin of constant thickness as depicted in Figure 17.10. The appropriate expressions for $A$ and $P$, in this case, are

$$A = 4\pi rt \quad r_0 \leq r \leq r_L$$

and

$$P = 4\pi r$$
When these expressions are substituted into equation (17-28), the applicable differential equation, considering $k$ and $h$ constant, is

$$
\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} - \frac{h}{kt} (T - T_\infty) = 0
$$

(17-33)

In each of the cases considered, the thermal conductivity and convective heat-transfer coefficient were assumed constant. When the variable nature of these quantities is considered, the resulting differential equations become still more complex than those developed thus far.

Governing Differential Equation for Circular Fin: Temperature variation in the R (radial) direction only!

$T = T(r)$
Figure 17.11 Fin efficiency for straight and circular fins of constant thickness.
The total heat transfer from a finned surface is

\[ q_{\text{total}} = q_{\text{primary surface}} + q_{\text{fin}} \]
\[ = A_0 h(T_0 - T_\infty) + A_f h(T - T_\infty) \]  \hspace{1cm} (17-47)

The second term in equation (17-47) is the actual heat transfer from the fin surface in terms of the variable surface temperature. This may be written in terms of the fin efficiency, yielding

\[ q_{\text{total}} = A_0 h(T_0 - T_\infty) + A_f h \eta_f (T_0 - T_\infty) \]

or

\[ q_{\text{total}} = h(A_0 + A_f \eta_f)(T_0 - T_\infty) \]  \hspace{1cm} (17-48)

In this expression, \( A_0 \) represents the exposed area of the primary surface, \( A_f \) is the total fin surface area, and the heat transfer coefficient, \( h \), is assumed constant.

The application of equation (17-48) as well as an idea of the effectiveness of fins is illustrated in Example 3.
Problem: Water and air are separated by a mild-steel plane wall. It is proposed to increase the heat-transfer rate between these fluids by adding straight rectangular fins of 1.27mm thickness, and 2.5-cm length, spaced 1.27 cm apart.

Note:
1. $100 \text{cm} / 1.27 \approx 79$ fins
2. $1 \text{m} \times 1 \text{m}$ is by free selection.
The fin surface area has only 3 sides.
   - Top: $(1 \text{m}) \times (0.025 \text{m})$
   - Bottom: $(1 \text{m}) \times (0.005 \text{m})$
   - Side: $0.00127 \times 1 \text{m}$

Total fin area = $79 \times 0.01 \text{m}^2 [2 \times 0.0025 \text{m}]$
               $+ 0.00127 \times 0.025 \text{m}^2$
               $= 0.15 \text{m}^2$

3. The fin efficiency is given by Figure 19.11, by calculating the x-axis coordinate:
   $x = (0.5 - x_0 \sqrt{h/k})$

   Full thickness rather than half thickness
The air-side and water-side heat-transfer coefficients may be assumed constant with values of 11.4 and 256 W/m²·K, respectively. Determine the percent change in total heat transfer when fins are placed on a) the water side, b) the air side, and c) both sides.

For a 1 m² section of the wall the areas of the primary surface and of the fins are

\[ A_o = 1 \text{ m}^2 - 79 \text{ fins (1 m)} \left[ \frac{0.00127 \text{ m}}{\text{fin}} \right] \]
\[ = 0.90 \text{ m}^2 \]
\[ A_f = 79 \text{ fins (1 m)} [(2)(0.025 \text{ m})] + 0.10 \text{ m}^2 \]
\[ = 4.05 \text{ m}^2 \]

Values of fin efficiency can now be determined from Figure 17.11. For the air side

\[ L\sqrt{h/k_t} = 0.025 \text{ m} \left[ \frac{11.4 \text{ W/m}^2 \cdot \text{K}}{(42.9 \text{ W/m} \cdot \text{K})(0.00127 \text{ m})} \right]^{1/2} \]
\[ = 0.362 \]

and for the water side

\[ L\sqrt{h/k_t} = 0.025 \text{ m} \left[ \frac{256 \text{ W/m}^2 \cdot \text{K}}{(42.9 \text{ W/m} \cdot \text{K})(0.00127 \text{ m})} \right]^{1/2} \]
\[ = 1.71 \]

The fin efficiencies are then read from the figure as

\[ \eta_{\text{air}} \approx 0.95 \]
\[ \eta_{\text{water}} \approx 0.55 \]

The total heat transfer rates can now be evaluated. For fins on the air side

\[ q = h_o \Delta T_o A_o + \eta_{ja} A_f \]
\[ = 11.4 \Delta T_o [0.90 + 0.95(4.05)] \]
\[ = 54.1 \Delta T_o \]
and on the water side

\[ q = h_w \Delta T_w \left| A_o + \eta_{fw} A_f \right| \]
\[ = 256 \Delta T_w [0.90 + 0.55(4.05)] \]
\[ = 801 \Delta T_w \]

The quantities \( \Delta T_a \) and \( \Delta T_w \) represent the temperature differences between the steel surface at temperature \( T_o \), and the fluids.

The reciprocals of the coefficients are the thermal resistances of the finned surfaces.

Without fins the heat-transfer rate in terms of the overall temperature difference, \( \Delta T = T_w - T_o \), neglecting the conductive resistance of the steel wall, is

\[ q = \frac{\Delta T}{\frac{1}{11.4} + \frac{1}{256}} = 10.91 \Delta T \]

With fins on the air side alone

\[ q = \frac{\Delta T}{\frac{1}{54.1} + \frac{1}{256}} = 44.67 \Delta T \]
With fins on the air side alone

\[ q = \frac{\Delta T}{\frac{1}{54.1} + \frac{1}{256}} = 44.67 \Delta T \]

an increase of 310 percent compared with the bare-wall case.

With fins on the water side alone

\[ q = \frac{\Delta T}{\frac{1}{11.4} + \frac{1}{801}} = 11.24 \Delta T \]

an increase of 3.0 percent.

With fins on both sides the heat-flow rate is

\[ q = \frac{\Delta T}{\frac{1}{54.1} + \frac{1}{801}} = 50.68 \Delta T \]

an increase of 365 percent.

This result indicates that adding fins is particularly beneficial where the convection coefficient has a relatively small value.