Week # 1
MR Chapter 1

• Tutorial #1
• MR #1.1, 1.4, 1.7.
• To be discussed on Jan. 24 2018.
• By either volunteer or class list.

MARTIN RHODES (2008) 
Publisher John Wiley & Son, Chichester, West Sussex, England.
• Describing the size of a single particle. Some terminology about diameters used in microscopy.

• Equivalent circle diameter.

• Martin’s diameter.

• Feret’s diameter.

• Shear diameter.
Describing the size of a single particle

• Regular-shaped particles

<table>
<thead>
<tr>
<th>Shape</th>
<th>Sphere</th>
<th>Cube</th>
<th>Cylinder</th>
<th>Cuboid</th>
<th>Cone</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensions</td>
<td>Radius</td>
<td>Side length</td>
<td>Radius and height</td>
<td>Three side lengths</td>
<td>Radius and height</td>
</tr>
</tbody>
</table>

• The orientation of the particle on the microscope slide will affect the projected image and consequently the measured equivalent sphere diameter.

• Sieve measurement: Diameter of a sphere passing through the same sieve aperture.

• Sedimentation measurement: Diameter of a sphere having the same sedimentation velocity under the same conditions.
Comparison of equivalent diameters

• The volume equivalent sphere diameter is a commonly used equivalent sphere diameter.
• Example: Coulter counter size measurement. The diameter of a sphere having the same volume as the particle.
• Surface-volume diameter is the diameter of a sphere having the same surface to volume ratio as the particle.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Comparison of equivalent sphere diameters</th>
<th>(Example)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cuboid</td>
<td>Sphere passing the same sieve aperture, $x_p$</td>
<td>Sphere having the same volume, $x_v$</td>
</tr>
<tr>
<td>Cylinder</td>
<td>3</td>
<td>3.06</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.38</td>
</tr>
</tbody>
</table>

Cuboid: side lengths of 1, 3, 5.
Cylinder: diameter 3 and length 1.
Description of populations of particles

\[ \frac{dF}{dx} = f(x) \]

F: Cumulative distribution, integral of the frequency distribution.
- Typical cumulative frequency distribution
• For a given population of particles, the distributions by mass, number and surface can differ dramatically.

• All are smooth continuous curves.

• Size measurement methods often divide the size spectrum into size ranges, and size distribution becomes a histogram.

• Comparison between distributions
Conversion between distributions

- Mass and number distributions for man-made objects orbiting the earth

<table>
<thead>
<tr>
<th>Size (cm)</th>
<th>Number of objects</th>
<th>% by number</th>
<th>% by mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>10–1000</td>
<td>7000</td>
<td>0.2</td>
<td>99.96</td>
</tr>
<tr>
<td>1–10</td>
<td>17 500</td>
<td>0.5</td>
<td>0.03</td>
</tr>
<tr>
<td>0.1–1.0</td>
<td>3 500 000</td>
<td>99.3</td>
<td>0.01</td>
</tr>
<tr>
<td>Total</td>
<td>3 524 500</td>
<td>100.00</td>
<td>100.00</td>
</tr>
</tbody>
</table>

Fraction of particles in the size range

\[ x \text{ to } x + dx = f_N(x)dx \]

Fraction of the total surface of particles in the size range

\[ x \text{ to } x + dx = f_S(x)dx \]
the fraction of the total surface area contained on these particles

\[ f_s dx = \frac{(x^2 \alpha_s)Nf_N(x)dx}{S} \]

- Total number of particles, \( N \) and total surface area \( S \) are constant.
- Particle shape is independent of size, \( \alpha_s \) is constant.

\[ f_s(x) \propto x^2f_N(x) \quad \text{or} \quad f_s(x) = k_Sx^2f_N(x) \]

\[ k_S = \frac{\alpha_S N}{S} \]

Similarly, for the distribution by volume

\[ f_v dx = \frac{(x^3 \alpha_v)Nf_N(x)dx}{V} \]

\[ f_V(x) = k_Vx^3f_N(x) \]

\[ k_V = \frac{\alpha_V N}{V} \]

\( V \) is the total volume of the particle population and \( \alpha_v \) is the factor relating the linear dimension of particle to its volume.
And for the distribution by mass

\[ f_m(x) = k_m, \quad x^3f_N(x) \]

where

\[ k_m = \frac{\alpha_V \rho_p N}{V} \]

assuming particle density \( \rho_p \) is independent of size.

The constants \( k_s, k_v \) and \( k_m \) may be found by using the fact that:

\[ \int_0^\infty f(x)dx = 1 \]
Assumptions for conversions among different distribution functions

• It is necessary to make assumptions about the constancy of shape and density with size.
• Calculation errors are introduced into the conversions.
• Example: 2% error in FN results in 6% error in FM. (Recalling the relationship between mass and diameter).
• If possible, direct measurements be made with the required distribution.
Describing the population by a single number

- Definitions of means

<table>
<thead>
<tr>
<th>$g(x)$</th>
<th>Mean and notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>arithmetic mean, $\bar{x}_a$</td>
</tr>
<tr>
<td>$x^2$</td>
<td>quadratic mean, $\bar{x}_q$</td>
</tr>
<tr>
<td>$x^3$</td>
<td>cubic mean, $\bar{x}_c$</td>
</tr>
<tr>
<td>$\log x$</td>
<td>geometric mean, $\bar{x}_g$</td>
</tr>
<tr>
<td>$1/x$</td>
<td>harmonic mean, $\bar{x}_h$</td>
</tr>
</tbody>
</table>

$$g(\bar{x}) = \frac{\int_0^1 g(x) \, dF}{\int_0^1 dF} \quad \text{but} \quad \int_0^1 dF = 1 \quad \text{and so} \quad g(\bar{x}) = \int_0^1 g(x) \, dF$$
• Plot of cumulative frequency against weighting function $g(x)$. Shaded area is $g(\bar{x}) = \int_{0}^{1} g(x) \, dF$

Number-length mean: Arithmetic mean of the number distribution conserves the number and length of population.

$$\bar{x}_{NL} = \bar{x}_{aN} = \frac{\int_{0}^{1} x \, dF_N}{\int_{0}^{1} dF_N}$$

Number-surface mean, $\bar{x}_{NS} = \bar{x}_{qN} = \frac{\int_{0}^{1} x^2 \, dF_N}{\int_{0}^{1} dF_N}$
• Comparison between measures of central tendency. Adapted from Rhodes (1990).

• Surface-volume mean, Sauter mean: Arithmetic mean of surface distribution conserves the surface and volume of population.

• The values of the different expressions of central tendency can vary significantly.

• Two quite different distributions could have the same arithmetic mean or median.
K_s and K_v do not vary with size

\[ dF_s = x^2 k_s dF_N \]
Common methods of displaying size distributions

Arithmetic-normal Distribution

$$\frac{dF}{dx} = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right]$$

Log-normal Distribution

$$\frac{dF}{dz} = \frac{1}{\sigma_z \sqrt{2\pi}} \exp \left[ -\frac{(z - \bar{z})^2}{2\sigma_z^2} \right]$$

$z = \log x$

$\bar{z}$: Arithmetic mean of $z$, $\sigma_z$: standard deviation of $\log x$

Arithmetic-normal distribution with an arithmetic mean of 45 and standard deviation of 12.
• Log-normal distribution plotted on linear coordinates

• Log-normal distribution plotted on logarithmic coordinates
Methods of particle size measurements: Sieving

• Sieving: Dry sieving using woven wire sieves is appropriate for particle size greater than 45 μm. The length of the particle does not hinder it passage through the sieve aperture.

• Most common modern sieves are in sizes such that the ratio of adjacent sieve sizes is the fourth root of two (e.g. 45, 53, 63, 75, 90, 107 μm).
Methods of particle size measurements: Microscopy

- The optical microscope may be used to measure particle size down to 5 μm.
- The electron microscope may be used for size analysis below 5 μm.
- Coupled with an image analysis system, the optical and electron microscopy can give **number** distribution of size and shape.
- For irregular-shaped particles, the projected area offered to the viewer can vary significantly. Technique (e.g. applying adhesive to the microscope slide) may be used to ensure “**random orientation**”.
Methods of particle size measurement

• **Sedimentation**

- \( \text{Re}_p < 0.3 \). Motion of the particle obeys Stoke’s law.
- The suspension is sufficiently dilute (No hindered settling).
- Particles are assumed to accelerate rapidly to their terminal free fall velocity, time for acceleration is negligible.

• **Size analysis by sedimentation**
$C_0$: original uniform suspension density.

Sampling point: $C$ at time $t$ after the start of settling.

At time $t$ all particles traveling faster than $h/t$ will have fallen below the sampling point.

$C$ represents the suspension density for all particles which travel at a velocity $\leq h/t$.

The diameter calculated from the Carman-Kozeny equation is the arithmetic mean of the surface distribution.

**Permeametry**

\[
U_T = \frac{x^2(\rho_p - \rho_f)g}{18 \mu}
\]

\[
x = \left[ \frac{18 \mu h}{t(\rho_p - \rho_f)g} \right]^{1/2}
\]

\[
\frac{(-\Delta p)}{H} = 180 \frac{(1 - \varepsilon)^2 \mu U}{\varepsilon^3 \frac{x^2}{H}}
\]

See Example 1.3

The diameter calculated from the Carman-Kozeny equation is the arithmetic mean of the surface distribution.
• **Electrozone sensing**

As particle flow through the orifice, a voltage pulse is recorded.

The amplitude of the pulse can be related to the volume of particle the orifice.

Particle range: 0.3-1000 μm.

• **Schematic of electrozone sensing apparatus**
WORKED EXAMPLE 1.1

Calculate the equivalent volume sphere diameter $x_v$ and the surface-volume equivalent sphere diameter $x_{sv}$ of a cuboid particle of side length 1, 2, 4 mm.

Solution

The volume of cuboid $= 1 \times 2 \times 4 = 8 \text{ mm}^3$

The surface area of the particle $= (1 \times 2) + (1 \times 2) + (1 + 2 + 1 + 2) \times 4 = 28 \text{ mm}^2$

The volume of sphere of diameter $x_v$ is $\frac{\pi x_v^3}{6}$

Hence, diameter of a sphere having a volume of $8 \text{ mm}^3$, $x_v = 2.481 \text{ mm}$

The equivalent volume sphere diameter $x_v$ of the cuboid particle is therefore $x_v = 2.481 \text{ mm}$

The surface to volume ratio of the cuboid particle $= \frac{28}{8} = 3.5 \text{ mm}^2/\text{mm}^3$

The surface to volume ratio for a sphere of diameter $x_{sv}$ is therefore $6/x_{sv}$

Hence, the diameter of a sphere having the same surface to volume ratio as the particle $= \frac{6}{3.5} = 1.714 \text{ mm}$

The surface-volume equivalent sphere diameter of the cuboid, $x_{sv} = 1.714 \text{ mm}$
WORKED EXAMPLE 1.2

Convert the surface distribution described by the following equation to a cumulative volume distribution:

\[ F_S = \left(\frac{x}{45}\right)^2 \quad \text{for} \quad x \leq 45 \mu m \]
\[ F_S = 1 \quad \text{for} \quad x > 45 \mu m \]

Solution

From Equations (1.1)–(1.3),

\[ f_v(x) = \frac{k_v}{k_s} x f_s(x) \]

Integrating between sizes 0 and \( x \):

\[ F_v(x) = \int_0^x \left(\frac{k_v}{k_s}\right) x f_s(x) \, dx \]

Noting that \( f_s(x) = \frac{dF_s}{dx} \), we see that

\[ f_s(x) = \frac{d}{dx} \left(\frac{x}{45}\right)^2 = \frac{2x}{(45)^2} \]
and our integral becomes

\[ F_v(x) = \int_0^x \left( \frac{k_v}{k_s} \right) \frac{2x^2}{(45)^2} \, dx \]

Assuming that \( k_v \) and \( k_s \) are independent of size,

\[ F_v(x) = \left( \frac{k_v}{k_s} \right) \frac{2x^3}{(45)^2} \, dx \]

\[ = \frac{2}{3} \left[ \frac{x^3}{(45)^2} \right] k_v \]

\( k_v / k_s \) may be found by noting that \( F_v(45) = 1 \); hence

\[ \frac{90k_v}{3k_s} = 1 \]  and so \[ \frac{k_v}{k_s} = 0.0333 \]

Thus, the formula for the volume distribution is

\[ F_v = 1.096 \times 10^{-5} x^3 \text{ for } x \leq 45 \mu m \]

\[ F_v = 1 \text{ for } x > 45 \mu m \]
WORKED EXAMPLE 1.3

What mean particle size do we use in calculating the pressure gradient for flow of a fluid through a packed bed of particles using the Carman–Kozeny equation (see Chapter 6)?

Solution

The Carman–Kozeny equation for laminar flow through a randomly packed bed of particles is:

$$\frac{(-\Delta p)}{L} = K \frac{(1 - \varepsilon)^2}{\varepsilon^3} S_v^2 \mu U$$

where $S_v$ is the specific surface area of the bed of particles (particle surface area per unit particle volume) and the other terms are defined in Chapter 6. If we assume that the bed voidage is independent of particle size, then to write the equation in terms of a mean particle size, we must express the specific surface, $S_v$, in terms of that mean. The particle size we use must give the same value of $S_v$ as the original population or particles. Thus the mean diameter $\bar{x}$ must conserve the surface and volume of the population; that is, the mean must enable us to calculate the total volume from the total surface of the particles. This mean is the surface-volume mean $\bar{x}_{sv}$

$$\bar{x}_{sv} \times \text{(total surface)} \times \frac{\alpha_v}{\alpha_s} = \text{(total volume)} \left(\text{eg. for spheres, } \frac{\alpha_v}{\alpha_s} = \frac{1}{6}\right)$$
and therefore $\bar{x}_{sv} \int_0^\infty f_s(x)dx \cdot \frac{k_v}{k_s} = \int_0^\infty f_v(x)dx$

Total volume of particles, $V = \int_0^\infty x^3\alpha_v N f_N(x)dx$

Total surface area of particles, $S = \int_0^\infty x^2\alpha_s N f_N(x)dx$

Hence, $\bar{x}_{sv} = \frac{\alpha_s \int_0^\infty x^3\alpha_v N f_N(x)dx}{\alpha_v \int_0^\infty x^2\alpha_s N f_N(x)dx}$

Then, since $\alpha_v$, $\alpha_s$ and $N$ are independent of size, $x$,

$\bar{x}_{sv} = \frac{\int_0^\infty x^3 f_N(x)dx}{\int_0^\infty x^2 f_N(x)dx} = \frac{\int_0^1 x^3 dF_N}{\int_0^1 x^2 dF_N}$

This is the definition of the mean which conserves surface and volume, known as the surface-volume mean, $\bar{x}_{sv}$. 
So

\[
\bar{x}_{SV} = \frac{\int_{0}^{1} x^3 \, dF_N}{\int_{0}^{1} x^2 \, dF_N}
\]  \hspace{1cm} (1.8)

The correct mean particle diameter is therefore the surface-volume mean as defined above. (We saw in Section 1.6 that this may be calculated as the arithmetic mean of the surface distribution \(\bar{x}_{as}\), or the harmonic mean of the volume distribution.) Then in the Carman–Kozeny equation we make the following substitution for \(S_v\):

\[
S_v = \frac{1}{\bar{x}_{SV}} \frac{k_s}{k_v}
\]

e.g. for spheres, \(S_v = 6/\bar{x}_{SV}\).
WORKED EXAMPLE 1.4 (AFTER SVAROVSKY, 1990)

A gravity settling device processing a feed with size distribution \( F(x) \) and operates with a grade efficiency \( G(x) \). Its total efficiency is defined as:

\[
E_T = \int_0^1 G(x) dF_M
\]

How is the mean particle size to be determined?

**Solution**

Assuming plug flow (see Chapter 3), \( G(x) = U_T A/Q \) where, \( A \) is the settling area, \( Q \) is the volume flow rate of suspension and \( U_T \) is the single particle terminal velocity for particle size \( x \), given by (in the Stokes region):

\[
U_T = \frac{x^2(\rho_p - \rho_f)g}{18 \mu} \tag{Chapter 2}
\]

hence

\[
E_T = \frac{Ag(\rho_p - \rho_f)}{18 \mu Q} \int_0^1 x^2 dF_M
\]

where \( \int_0^1 x^2 dF_M \) is seen to be the definition of the quadratic mean of the distribution by mass \( \bar{x}_{qM} \) (see Table 1.4).
WORKED EXAMPLE 1.5

A Coulter counter analysis of a cracking catalyst sample gives the following cumulative volume distribution:

<table>
<thead>
<tr>
<th>Channel</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>% volume differential</td>
<td>0</td>
<td>0.5</td>
<td>1.0</td>
<td>1.6</td>
<td>2.6</td>
<td>3.8</td>
<td>5.7</td>
<td>8.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Channel</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>% volume differential</td>
<td>14.3</td>
<td>22.2</td>
<td>33.8</td>
<td>51.3</td>
<td>72.0</td>
<td>90.9</td>
<td>99.3</td>
<td>100</td>
</tr>
</tbody>
</table>

(a) Plot the cumulative volume distribution versus size and determine the median size.

(b) Determine the surface distribution, giving assumptions. Compare with the volume distribution.

(c) Determine the harmonic mean diameter of the volume distribution.

(d) Determine the arithmetic mean diameter of the surface distribution.
Solution

With the Coulter counter the channel size range differs depending on the tube in use. We therefore need the additional information that in this case channel 1 covers the size range 3.17 μm to 4.0 μm, channel 2 covers the range 4.0 μm to 5.04 μm and so on up to channel 16, which covers the range 101.4 μm to 128 μm. The ratio of adjacent size range boundaries is always the cube root of 2. For example,

\[
\sqrt[3]{\frac{5.04}{4.0}} = \frac{128}{101.4}, \text{ etc.}
\]

The resulting lower and upper sizes for the channels are shown in columns 2 and 3 of Table 1W5.1.

![Figure 1W5.1 Cumulative volume distribution](image)
Table 1W5.1  Size distribution data associated with Worked Example 1.5

<table>
<thead>
<tr>
<th>Channel number</th>
<th>Lower size of range μm</th>
<th>Upper size of range μm</th>
<th>Cumulative per cent undersize</th>
<th>$F_v$</th>
<th>$1/x$</th>
<th>Cumulative area under $F_v$ versus $1/x$</th>
<th>$F_s$</th>
<th>Cumulative area under $F_s$ versus $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.17</td>
<td>4.00</td>
<td>0</td>
<td>0</td>
<td>0.2500</td>
<td>0.00000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>4.00</td>
<td>5.04</td>
<td>0.5</td>
<td>0.005</td>
<td>0.1984</td>
<td>0.0011</td>
<td>0.0403</td>
<td>0.1823</td>
</tr>
<tr>
<td>3</td>
<td>5.04</td>
<td>6.35</td>
<td>1</td>
<td>0.01</td>
<td>0.1575</td>
<td>0.0020</td>
<td>0.0723</td>
<td>0.3646</td>
</tr>
<tr>
<td>4</td>
<td>6.35</td>
<td>8.00</td>
<td>1.6</td>
<td>0.016</td>
<td>0.1250</td>
<td>0.0029</td>
<td>0.1028</td>
<td>0.5834</td>
</tr>
<tr>
<td>5</td>
<td>8.00</td>
<td>10.08</td>
<td>2.6</td>
<td>0.026</td>
<td>0.0992</td>
<td>0.0040</td>
<td>0.1432</td>
<td>0.9480</td>
</tr>
<tr>
<td>6</td>
<td>10.08</td>
<td>12.70</td>
<td>3.8</td>
<td>0.038</td>
<td>0.0787</td>
<td>0.0050</td>
<td>0.1816</td>
<td>1.3855</td>
</tr>
<tr>
<td>7</td>
<td>12.70</td>
<td>16.00</td>
<td>5.7</td>
<td>0.057</td>
<td>0.0625</td>
<td>0.0064</td>
<td>0.2299</td>
<td>2.0782</td>
</tr>
<tr>
<td>8</td>
<td>16.00</td>
<td>20.16</td>
<td>8.7</td>
<td>0.087</td>
<td>0.0496</td>
<td>0.0081</td>
<td>0.2904</td>
<td>3.1720</td>
</tr>
<tr>
<td>9</td>
<td>20.16</td>
<td>25.40</td>
<td>14.3</td>
<td>0.143</td>
<td>0.0394</td>
<td>0.0106</td>
<td>0.3800</td>
<td>5.2138</td>
</tr>
<tr>
<td>10</td>
<td>25.40</td>
<td>32.00</td>
<td>22.2</td>
<td>0.222</td>
<td>0.0313</td>
<td>0.0134</td>
<td>0.4804</td>
<td>8.0942</td>
</tr>
<tr>
<td>11</td>
<td>32.00</td>
<td>40.32</td>
<td>33.8</td>
<td>0.338</td>
<td>0.0248</td>
<td>0.0166</td>
<td>0.5973</td>
<td>12.3236</td>
</tr>
<tr>
<td>12</td>
<td>40.32</td>
<td>50.80</td>
<td>51.3</td>
<td>0.513</td>
<td>0.0197</td>
<td>0.0205</td>
<td>0.7374</td>
<td>18.7041</td>
</tr>
<tr>
<td>13</td>
<td>50.80</td>
<td>64.00</td>
<td>72</td>
<td>0.72</td>
<td>0.0156</td>
<td>0.0242</td>
<td>0.8689</td>
<td>26.2514</td>
</tr>
<tr>
<td>14</td>
<td>64.00</td>
<td>80.63</td>
<td>90.9</td>
<td>0.909</td>
<td>0.0124</td>
<td>0.0268</td>
<td>0.9642</td>
<td>33.1424</td>
</tr>
<tr>
<td>15</td>
<td>80.63</td>
<td>101.59</td>
<td>99.3</td>
<td>0.993</td>
<td>0.0098</td>
<td>0.0277</td>
<td>0.9978</td>
<td>36.2051</td>
</tr>
<tr>
<td>16</td>
<td>101.59</td>
<td>128.00</td>
<td>100</td>
<td>1</td>
<td>0.0078</td>
<td><strong>0.0278</strong></td>
<td>1.0000</td>
<td><strong>36.4603</strong></td>
</tr>
</tbody>
</table>

Note: Based on arithmetic means of size ranges.
(a) The cumulative undersize distribution is shown numerically in column 5 of Table 1W5.1 and graphically in Figure 1W5.1. By inspection, we see that the median size is 50 µm (b), i.e. 50% by volume of the particles is less than 50 µm.

(b) The surface distribution is related to the volume distribution by the expression:

\[ f_s(x) = \frac{f_v(x)}{x} \times \frac{k_s}{k_v} \quad \text{(from [Equations (1.1) and (1.2)])} \]

Recalling that \( f(x) = \frac{dF}{dx} \) and integrating between 0 and \( x \):

\[ \frac{k_s}{k_v} \int_0^x \frac{1}{x} \frac{dF_v}{dx} \, dx = \int_0^x \frac{dF_s}{dx} \, dx \]

or

\[ \frac{k_s}{k_v} \int_0^x \frac{1}{x} \, dF_v = \int_0^x dF_s = F_s(x) \]

(assuming particle shape is invariant with size so that \( k_s/k_v \) is constant).

So the surface distribution can be found from the area under a plot of \( 1/x \) versus \( F_v \) multiplied by the factor \( k_s/k_v \) (which is found by noting that \( \int_{x=0}^{x=\infty} dF_s = 1 \)).
Column 7 of Table 1W5.1 shows the area under $1/x$ versus $F_v$. The factor $k_s/k_v$ is therefore equal to 0.0278. Dividing the values of column 7 by 0.0278 gives the surface distribution $F_s$ shown in column 8. The surface distribution is shown graphically in Figure 1W5.2. The shape of the surface distribution is quite different from that of the volume distribution; the smaller particles make up a high proportion of the total surface. The median of the surface distribution is around 35 μm, i.e. particles under 35 μm contribute 50% of the total surface area.

(c) The harmonic mean of the volume distribution is given by:

$$\frac{1}{\bar{x}_{hV}} = \int_0^1 \left(\frac{1}{x}\right) dF_v$$

![Diagram](image.png)

**Figure 1W5.2** Cumulative surface distribution
This can be calculated graphically from a plot of $F_v$ versus $1/x$ or numerically from the tabulated data in column 7 of Table 1W5.1. Hence,

$$\frac{1}{\bar{x}_{hV}} = \int_0^1 \left(\frac{1}{x}\right) dF_v = 0.0278$$

and so, $\bar{x}_{hV} = 36 \mu m$.

We recall that the harmonic mean of the volume distribution is equivalent to the surface-volume mean of the population.

(d) The arithmetic mean of the surface distribution is given by:

$$\bar{x}_{as} = \int_0^1 x \, dF_s$$

This may be calculated graphically from our plot of $F_s$ versus $x$ (Figure 1W5.2) or numerically using the data in Table 1W5.1. This area calculation as shown in Column 9 of the table shows the cumulative area under a plot of $F_s$ versus $x$ and so the last figure in this column is equivalent to the above integral.

Thus:

$$\bar{x}_{as} = 36.4 \mu m$$

We may recall that the arithmetic mean of the surface distribution is also equivalent to the surface-volume mean of the population. This value compares well with the value obtained in (c) above.
WORKED EXAMPLE 1.6

Consider a cuboid particle $5.00 \times 3.00 \times 1.00$ mm. Calculate for this particle the following diameters:

(a) the volume diameter (the diameter of a sphere having the same volume as the particle);

(b) the surface diameter (the diameter of a sphere having the same surface area as the particle);

(c) the surface-volume diameter (the diameter of a sphere having the same external surface to volume ratio as the particle);

(d) the sieve diameter (the width of the minimum aperture through which the particle will pass);

(e) the projected area diameters (the diameter of a circle having the same area as the projected area of the particle resting in a stable position).

Solution

(a) Volume of the particle $= 5 \times 3 \times 1 = 15$ mm$^3$

Volume of a sphere $= \frac{\pi x_v^3}{6}$
Thus volume diameter, $x_v = \sqrt[3]{\frac{15 \times 6}{\pi}} = 3.06 \text{ mm}$

(b) Surface area of the particle $= 2 \times (5 \times 3) + 2 \times (1 \times 3) + 2 \times (1 \times 5) = 46 \text{ mm}^2$

Surface area of sphere $= \pi x_s^2$

Therefore, surface diameter, $x_s = \sqrt{\frac{46}{\pi}} = 3.83 \text{ mm}$

(c) Ratio of surface to volume of the particle $= \frac{46}{15} = 3.0667$

For a sphere, surface to volume ratio $= \frac{6}{x_{sv}}$

Therefore, $x_{sv} = \frac{6}{3.0667} = 1.96 \text{ mm}$

(d) The smallest square aperture through which this particle will pass is 3 mm. Hence, the sieve diameter, $x_p = 3 \text{ mm}$

(e) This particle has three projected areas in stable positions:

\[
\begin{align*}
\text{area 1} &= 3 \text{ mm}^2; \\
\text{area 2} &= 5 \text{ mm}^2; \\
\text{area 3} &= 15 \text{ mm}^2
\end{align*}
\]

area of circle $= \frac{\pi x^2}{4}$

hence, projected area diameters:

- projected area diameter 1 = 1.95 mm;
- projected area diameter 2 = 2.52 mm;
- projected area diameter 3 = 4.37 mm.