Week # 3
MR Chapter 3

- Tutorial #3
- MR #3.1, 3.5, 3.9, 3.12.
- To be discussed on Feb 7, 2018.
- By either volunteer or class list.
Settling of a suspension of particles

effective viscosity, $\mu_e = \mu / f(\varepsilon)$

average suspension density, $\rho_{ave} = \varepsilon \rho_f + (1 - \varepsilon) \rho_p$

$$C_D = \frac{24}{\text{Re}_p} = \frac{24\mu_e}{U_{rel}\rho_{ave} \chi}$$

$$C_D = R' / \left( \frac{1}{2} \rho_{ave} U_{rel}^2 \right)$$

drag force = weight − upthrust

$$\left( \frac{\pi x^2}{4} \right) \frac{1}{2} \rho_{ave} U_{rel}^2 C_D = (\rho_p - \rho_{ave}) \left( \frac{\pi x^3}{6} \right) g$$

$$U_{rel} = (\rho_p - \rho_{ave}) \frac{x^2 g}{18 \mu_e}$$

Equation (3.5)
\[ U_{relT} = (\rho_p - \rho_f) \frac{x^2 g}{18 \mu} e_f(\varepsilon) \quad U_{relT} = U_T e_f(\varepsilon) \]

Equation (3.6)

Superficial fluid velocity, \( U_{fs} = \frac{Q_f}{A} \)

Superficial particle velocity, \( U_{ps} = \frac{Q_p}{A} \)

Flow area occupied by the fluid, \( A_f = \varepsilon A \)

Flow area occupied by the particles, \( A_p = (1 - \varepsilon)A \)

For the fluid: \( Q_f = U_{fs} A = U_f A \varepsilon \)

For the particles: \( Q_p = U_{ps} A = U_p A (1 - \varepsilon) \)

Actual velocity of the fluid, \( U_f = \frac{U_{fs}}{\varepsilon} \)

Actual velocity of the particles, \( U_p = \frac{U_{ps}}{(1 - \varepsilon)} \)
Batch settling

\[ Q_p + Q_f = 0 \]

\[ U_p(1 - \varepsilon) + U_f\varepsilon = 0 \]

\[ U_f = -U_p \frac{(1 - \varepsilon)}{\varepsilon} \]

\[ U_p = U_T \varepsilon^2 f(\varepsilon) \]

\[ f(\varepsilon) = \varepsilon^{2.5} \]

Equation (3.20)

Re < 0.3:

\[ U_p = U_T \varepsilon^{4.65} \text{ [giving } f(\varepsilon) = \varepsilon^{2.65}] \]

Re > 500:

\[ U_p = U_T \varepsilon^{2.4} \text{ [giving } f(\varepsilon) = \varepsilon^{0.4}] \]
Khan & Richardson

Archimedes number, Ar:

\[ \frac{4.8 - n}{n - 2.4} = 0.043 Ar^{0.57} \left[ 1 - 2.4 \left( \frac{x}{D} \right)^{0.27} \right] \]

\[ U_p = U_T \varepsilon^n \]

\[ \frac{U_{ps}}{U_T} = (1 - \varepsilon) \varepsilon^n \quad (*) \]

Equation (*): A maximum is observed at \( \varepsilon = (n/(n+1)) \);
Inflection point at \( \varepsilon = (n-1)/(n+1) \)
• Sharp interfaces in sedimentation

Variation of dimensionless setting flux with suspension concentration

$Re_p < 0.3, \ n = 4.65$
Concentration interface in sedimentation

\[ (U_{p1} - U_{int})C_1 = (U_{p2} - U_{int})C_2 \]

\[ U_{int} = \frac{U_{ps1} - U_{ps2}}{C_1 - C_2} \]

\[ U_{int} = \frac{\Delta U_{ps}}{\Delta C} \]

\[ U_{int} = \frac{U_{p1}C_1 - U_{p2}C_2}{C_1 - C_2} \]

\[ U_{ps} = U_p \cdot C = U_p \cdot (1-\varepsilon) \]

(a) The gradient of the curve at concentration C is the velocity of a layer of suspension of this concentration.

(b) The slope of a chord joining two points at concentration \( C_1 \) and \( C_2 \) is the velocity of discontinuity or interface between suspensions of these concentrations.
The batch settling test

Determination of interface and layer velocities from a batch flux plot
Type 1 batch settling. Zones A, B and S are zones of constant concentration. Zone A is a clear liquid; zone B is a suspension of concentration equal to the initial suspension concentration; zone S is a suspension of settled bed or sediment concentration.
Change in positions of interface AB, BS and AS with time in Type 1 batch settling
Type 2 batch settling. Zones A, B and S are zones of constant concentration. Zone A is clear liquid; zone B is a suspension of concentration equal to the initial suspension concentration; zone S is a suspension of settled bed concentration. Zone E is a zone of variable concentration.
Change in positions of interface AB, BE_{min}, E_{max} S and AS with time in Type 2 batch settling
Determining if settling will be Type 1 or Type 2. A line through $C_S$ tangent to the flux curve gives $C_{B1}$ and $C_{B2}$. Type 2 settling occurs when initial suspension concentration is between $C_{B1}$ and $C_{B2}$. 
• Relationship between the Height-Time curve and the flux plot

\[
\text{velocity of interface} = \frac{\text{dh}}{\text{dt}} = \frac{h_1 - h}{t}
\]

Analysis of batch settling test
Analysis of batch settling; relative velocities of a plane of concentration \( C \) and the particles in the plane

\[
U_p = \frac{h_1 - h}{t}
\]
velocity of particles relative to plane = \( U_p + \frac{h}{t} \)

Volume of particles which have passed through this plane in time \( t \)

\[
= \text{area} \times \text{velocity of particles} \times \text{concentration} \times \text{time}
\]

\[
= A \left( U_p + \frac{h}{t} \right) C t
\]

The total volume of all the particles in the test = \( C_B h_0 A \)

\[
C_B h_0 A = A \left( U_p + \frac{h}{t} \right) C t
\]

\[
C = \frac{C_B h_0}{h_1}
\]
Continuous settling

- Settling of a suspension in a flowing fluid

\[ Q = (U_{ps} + U_{fs})A \]
Upward flow of a particle suspension in a vessel gives total downward particle flux.

\[ U_{rel} = \frac{U_{ps}}{1 - \varepsilon} - \frac{U_{fs}}{\varepsilon} \]

\[ U_{rel} = U_T \varepsilon f(\varepsilon) \]

\[ U_{ps} = \frac{Q(1 - \varepsilon)}{A} + U_T \varepsilon^2 (1 - \varepsilon) f(\varepsilon) \]

** Equation (3.41) **

** Upward flow of a particle suspension in a vessel gives total downward particle flux. **

\[ U_{ps} = U_T \varepsilon^2 f(\varepsilon) - \frac{Q(1 - \varepsilon)}{A} \]

** total solids flux = flux due to settling – flux due to bulk flow **
Total flux plot for settling in **downward flow**
Total flux plot for settling in **upward flow**
• A real thickener (with upflow and downflow sections)

\[ F = V + L \]

A real thickener, combining upflow and downflow (F, L and V are volumes; \( C_F \), \( C_L \) and \( C_V \) are concentrations)

\[ FC_F = VC_V + LC_L \]
Total flux plot for a thickener at critical loading
Total flux plot for an underloaded thickener
Total flux plot for an overloaded thickener.
Alternative total flux plot shape; thickener at critical loading
Alternative total flux plot shape; Overloaded thickener
WORKED EXAMPLE 3.1

A height–time curve for the sedimentation of a suspension, of initial suspension concentration 0.1, in vertical cylindrical vessel is shown in Figure 3W1.1. Determine:

(a) the velocity of the interface between clear liquid and suspension of concentration 0.1;

(b) the velocity of the interface between clear liquid and a suspension of concentration 0.175;

(c) the velocity at which a layer of concentration 0.175 propagates upwards from the base of the vessel;

(d) the final sediment concentration.

Solution

(a) Since the initial suspension concentration is 0.1, the velocity required in this question is the velocity of the AB interface. This is given by the slope of the straight portion of the height–time curve.

\[
\text{Slope} = \frac{20 - 40}{15 - 0} = 1.333 \text{ cm/s}
\]
Figure 3W1.1  Batch settling test; height–time curve

(b) We must first find the point on the curve corresponding to the point at which a suspension of concentration 0.175 interfaces with the clear suspension. From Equation (3.38), with $C = 0.175$, $C_B = 0.1$ and $h_0 = 40 \text{ cm}$, we find:

$$h_1 = \frac{0.1 \times 40}{0.175} = 22.85 \text{ cm}$$
A line drawn through the point \( t = 0, h = h_1 \) tangent to the curve locates the point on the curve corresponding to the time at which a suspension of concentration 0.175 interfaces with the clear suspension (Figure 3W1.2). The coordinates of this point are \( t = 26 \text{ s}, h = 15 \text{ cm} \). The velocity of this interface is the slope of the curve at this point:

\[
\text{slope of curve at } 26 \text{ s}, 15 \text{ cm} = \frac{15 - 22.85}{26 - 0} = -0.302 \text{ cm/s}
\]

downward velocity of interface = 0.30 cm/s

(c) From the consideration above, after 26 s the layer of concentration 0.175 has just reached the clear liquid interface and has travelled a distance of 15 cm from the base of the vessel in this time.

Therefore, upward propagation velocity of this layer = \( \frac{h}{t} = \frac{15}{26} = 0.577 \text{ cm/s} \)

(d) To find the concentration of the final sediment we again use Equation (3.38). The value of \( h_1 \) corresponding to the final sediment \( (h_{1S}) \) is found by drawing a tangent to the part of the curve corresponding to the final sediment and projecting it to the \( h \) axis.

In this case \( h_{1S} = 10 \text{ cm} \) and so from Equation (3.38),

\[
\text{final sediment concentration, } C = \frac{C_0 h_0}{h_{1S}} = \frac{0.1 \times 4.0}{10} = 0.4
\]
Figure 3W1.2  Batch settling test
WORKED EXAMPLE 3.2

A suspension in water of uniformly sized sphere (diameter 150 μm, density 1140 kg/m³ has a solids concentration of 25% by volume. The suspension settles to a bed of solids concentration of 55% by volume. Calculate:

(a) the rate at which the water/suspension interface settles;

(b) the rate at which the sediment/suspension interface rises (assume water properties: density, 1000 kg/m³; viscosity, 0.001 Pa s).

Solution

(a) Solids concentration of initial suspension, \( C_B = 0.25 \)

Equation (3.28) allows us to calculate the velocity of interfaces between suspensions of different concentrations.

The velocity of the interface between initial suspension (B) and clear liquid (A) is therefore:

\[
U_{\text{int},AB} = \frac{U_{pA}C_A - U_{pB}C_B}{C_A - C_B}
\]

Since \( C_A = 0 \), the equation reduces to

\[
U_{\text{int},AB} = U_{pB}
\]
\( U_{PB} \) is the hindered settling velocity of particles relative to the vessel wall in batch settling and is given by Equation (3.24):

\[
U_p = U_T e^n
\]

Assuming Stokes' law applies, then \( n = 4.65 \) and the single particle terminal velocity is given by Equation (2.13) (see Chapter 2):

\[
U_T = \frac{x^2(\rho_p - \rho_f)g}{18 \mu}
\]

\[
U_T = \frac{9.81 \times (150 \times 10^{-6})^2 \times (1140 - 1000)}{18 \times 0.001}
\]

\[
= 1.717 \times 10^{-3} \text{ m/s}
\]

To check that the assumption of Stokes' law is valid, we calculate the single particle Reynolds number:

\[
Re_p = \frac{(150 \times 10^{-6}) \times 1.717 \times 10^{-3} \times 1000}{0.001}
\]

\[
= 0.258, \text{ which is less than the limiting value for Stokes' law (0.3) and so the assumption is valid.}
\]

The voidage of the initial suspension, \( \varepsilon_B = 1 - C_B = 0.75 \)

hence, \( U_{PB} = 1.717 \times 10^{-3} \times 0.75^{4.65} \)

\[
= 0.45 \times 10^{-3} \text{ m/s}
\]
Hence, the velocity of the interface between the initial suspension and the clear liquid is 0.45 mm/s. The fact that the velocity is positive indicates that the interface is moving downwards.

(b) Here again we apply Equation (3.28) to calculate the velocity of interfaces between suspensions of different concentrations.

The velocity of the interface between initial suspension (B) and sediment (S) is therefore

\[ U_{\text{int, BS}} = \frac{U_{PB}C_B - U_{PS}C_S}{C_B - C_S} \]

With \( C_B = 0.25 \) and \( C_S = 0.55 \) and since the velocity of the sediment, \( U_{PS} \) is zero, we have:

\[ U_{\text{int, BS}} = \frac{U_{PB}0.25}{0.25 - 0.55} = -0.833U_{PB} \]

And from part (a), we know that \( U_{PB} = 0.45 \text{mm/s} \), and so \( U_{\text{int, BS}} = -0.375 \text{mm/s} \).

The negative sign signifies that the interface is moving upwards. So, the interface between initial suspension and sediment is moving upwards at a velocity of 0.375 mm/s.
WORKED EXAMPLE 3.3

For the batch flux plot shown in Figure 3W3.1, the sediment has a solids concentration of 0.4 volume fraction of solids.

Figure 3W3.1  Batch flux plot
(a) Determine the range of initial suspension concentrations over which a zone of variable concentration is formed under batch settling conditions.

(b) Calculate and plot the concentration profile after 50 min in a batch settling test of a suspension with an initial concentration 0.1 volume fraction of solids, and initial suspension height of 100 cm.

(c) At what time will the settling test be complete?

Solution

(a) Determine the range of initial suspension concentrations by drawing a line through the point \( C = C_S = 0.4, U_{ps} = 0 \) tangent to the batch flux curve. This is shown as line \( XC_S \) in Figure 3W3.2. The range of initial suspension concentrations for which a zone of variable concentration is formed in batch settling (Type 2 settling) is defined by \( C_{B_{min}} \) and \( C_{B_{max}} \). \( C_{B_{min}} \) is the value of \( C \) at which the line \( XC_S \) intersects the settling curve and \( C_{B_{max}} \) is the value of \( C \) at the tangent. From Figure 3W3.2, we see that \( C_{B_{min}} = 0.036 \) and \( C_{B_{max}} = 0.21 \).

(b) To calculate the concentration profile we must first determine the velocities of the interfaces between the zones A, B, E and S and hence find their positions after 50 min.

The line AB in Figure 3W3.2 joins the point representing A the clear liquid (0, 0) and the point B representing the initial suspension (0.1, \( U_{ps} \)). The slope of line AB is equal to the
velocity of the interface between zones A and B. From Figure 3W3.2, \( U_{\text{int,AB}} = +0.166 \text{ mm/s or } +1.00 \text{ cm/min.} \)

The slope of the line from point B tangent to the curve is equal to the velocity of the interface between the initial suspension B and the minimum value of the variable concentration zone \( C_{E_{\text{min}}} \).

From Figure 3W3.2,

\[
U_{\text{int,BE}_{\text{min}}} = -0.111 \text{ mm/s or } -0.66 \text{ cm/min}
\]

The slope of the line tangent to the curve and passing through the point representing the sediment (point \( C = C_s = 0.4, U_{ps} = 0 \)) is equal to the velocity of the interface between the maximum value of the variable concentration zone \( C_{E_{\text{max}}} \) and the sediment.

From Figure 3W3.2,

\[
U_{\text{int,E}_{\text{maxS}}} = -0.0355 \text{ mm/s or } -0.213 \text{ cm/min}
\]

Therefore, after 50 min the distances travelled by the interfaces will be:

- **AB interface** 50.0 cm (1.00 x 50) downwards
- **BE_{\text{min}}** interface 33.2 cm upwards
- **E_{\text{maxS}}** interface 10.6 cm upwards

Therefore, the positions of the interfaces (distance from the base of the test vessel) after 50 min will be

- **AB interface** 50.0 cm
- **BE_{\text{min}}** interface 33.2 cm
- **E_{\text{maxS}}** interface 10.6 cm
Figure 3W3.2  Graphical solution to batch settling problem in Worked Example 3.3
From Figure 3W3.2 we determine the minimum and maximum values of suspension concentration in the variable zone

\[ C_{E_{\text{min}}} = 0.16 \]
\[ C_{E_{\text{max}}} = 0.21 \]

Using this information we can plot the concentration profile in the test vessel 50 min after the start of the test. A sketch of the profile is shown in Figure 3W3.3. The shape of the concentration profile within the variable concentration zone may be determined by the following method. Recalling that the slope of the batch flux plot (Figure 3W3.1) at a value of suspension concentration \( C \) is the velocity of a layer of suspension of that concentration, we find the slope at two or more values of concentration and then determine the positions of these layers after 50 min:

- Slope of batch flux plot at \( C = 0.18 \) is 0.44 cm/min upwards.

  Hence, position of a layer of concentration 0.18 after 50 min is 22.0 cm from the base.

- Slope of batch flux plot at \( C = 0.20 \) is 0.27 cm/min upwards.

  Hence, position of a layer of concentration 0.20 after 50 min is 13.3 cm from the base.

These two points are plotted on the concentration profile in order to determine the shape of the profile within the zone of variable concentration.

Figure 3W3.4 is a sketched plot of the height–time curve for this test constructed from the information above. The shape of the curved portion of the curve can again be
Figure 3W3.3  Sketch of concentration profile in batch settling test vessel after 50 min
Figure 3W3.4 Sketch of height–time curve for the batch settling test in Worked Example 3.3
determined by plotting the positions of two or more layers of suspension of different concentration. The initial suspension concentration zone (B) ends when the AB line intersects the BE_{min} line, both of which are plotted from a knowledge of their slopes.

The time for the end of the test is found in the following way. The end of the test is when the position of the E_{max}S interface coincides with the height of the final sediment. The height of the final sediment may be found using Equation (3.38) [see part (d) of Worked Example 3.1]:

\[ C_S h_S = C_B h_0 \]

where \( h_S \) is the height of the final sediment and \( h_0 \) is the initial height of the suspension (at the start of the test). With \( C_S = 0.4 \), \( C_B = 0.1 \) and \( h_0 = 100 \text{ cm} \), we find that \( h_S = 25 \text{ cm} \). Plotting \( h_S \) on Figure 3W3.4, we find that the E_{max}S line intersects the final sediment line at about 120 min and so the test ends at this time.
WORKED EXAMPLE 3.4

Using the flux plot shown in Figure 3W4.1:

(a) Graphically determine the limiting feed concentration for a thickener of area 100 m$^2$ handling a feed rate of 0.019 m$^3$/s and an underflow rate of 0.01 m$^3$/s. Under these conditions what will be the underflow concentration and the overflow concentration?

(b) Under the same flow conditions as above, the feed concentration is increased to 0.2. Estimate the solids concentration in the overflow, in the underflow, in the upflow section and in the downflow section of the thickener.

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**Figure 3W4.1** Batch flux plot
Solution

(a) Feed rate, $F = 0.019 \text{ m}^3/\text{s}$

Underflow rate, $L = 0.01 \text{ m}^3/\text{s}$

Material balance gives, overflow rate, $V = F - L = 0.009 \text{ m}^3/\text{s}$

Expressing these flows as fluxes based on the thickener area ($A = 100 \text{ m}^2$):

$$\frac{F}{A} = 0.19 \text{ mm/s}$$
$$\frac{L}{A} = 0.10 \text{ mm/s}$$
$$\frac{V}{A} = 0.09 \text{ mm/s}$$

The relationships between bulk flux and suspension concentration are then:

Feed flux = $C_F \left(\frac{F}{A}\right)$

Flux in underflow = $C_L \left(\frac{L}{A}\right)$

Flux in overflow = $C_V \left(\frac{V}{A}\right)$

Lines of slope $F/A$, $L/A$ and $-V/A$ drawn on the flux plot represent the fluxes in the feed, underflow and overflow, respectively (Figure 3W4.2). The total flux plot for the section below the feed point is found by adding the batch flux plot to the underflow flux line. The total flux plot for the section above the feed point is found by adding the batch flux plot to the overflow flux line.
Figure 3W4.2  Total flux plot: solution to part (a) of Worked Example 3.4
Figure 3W4.3  Solution to part (b) of Worked Example 3.4
flux plot to the overflow flux line (which is negative since it is an upward flux). These plots are shown in Figure 3W4.2.

The critical feed concentration is found where the feed flux line intersects the plot of total flux in the section below the feed (Figure 3.W4.2). This gives a critical feed flux of 0.0335 mm/s. The downflow section below the feed point is unable to take a flux greater than this. The corresponding feed concentration is $C_{F_{\text{crit}}} = 0.174$.

The concentration in the downflow section, $C_{B}$ is also 0.174.

The corresponding concentration in the underflow is found where the critical flux line intersects the underflow flux line. This gives $C_{L} = 0.33$.

(b) Referring now to Figure 3W4.3, if the feed flux is increased to 0.2, we see that the corresponding feed flux is 0.038 mm/s. At this feed concentration the downflow section is only able to take a flux of 0.034 mm/s and gives an underflow concentration, $C_{L} = 0.34$. The excess flux of 0.004 mm/s passes into the upflow section. This flux in the upflow section gives a concentration, $C_{T} = 0.2$ and a corresponding concentration, $C_{V} = 0.044$ in the overflow.