Temporal Aggregation, Causality Distortions, and a Sign Rule

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Abstract

Temporal aggregation creates contemporaneous correlations, alters dynamic links and may distort causality inference. Obtaining temporally disaggregated data series with an observation frequency that coincides with the causal lag is impractical in many situations. Since cointegration is invariant to temporal aggregation and implies Granger-causality this paper proposes a sign rule to establish the direction of causality. Temporal aggregation leads to a distortion of the sign of the adjustment coefficients of an error correction model. The sign rule helps in determining causality in levels and establishing weak exogeneity, which provide information for contemporaneous conditioning in a VECM or a structural VAR model. The sign rule is illustrated through three applications.

Key Words: cointegration, causality in levels, weak exogeneity, error correction model, adjustment coefficient, sign rule

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1. Introduction

Inconceivably contradictory results produced by Granger causality tests led Pagan (1989) to throw away baby with bathwater. The fragility of causal inference emanates mainly from two sources, (i) the information set used and (ii) temporal aggregation. The former is a problem that all econometric models have to deal with and the latter requires devising suitable statistical techniques that produce robust inferences regardless of the level of temporal aggregation.

A number of theoretical studies have established that temporal aggregation leads to misleading inference on Granger-causality (see Wei, 1990 and Marcellino, 1999 and references therein). Gulasekaran and Abeysinghe (2002) and Gulasekaran (2004) have derived quantitative results using an analytical framework to assess the nature of the distortions created. Overall the following conclusions emerge. Within a stationary framework temporal aggregation may (i) create a spurious feedback loop from a unidirectional relation, (ii) erase a feedback loop and create a unidirectional relation and (iii) erase the Granger-causal link altogether. The distortions magnify when differencing is used after temporal aggregation to induce stationarity.

These findings of distortions are not much of a comfort in practice because most available data series are either temporally aggregated or systematically sampled depending on whether the variables are flows or stocks respectively. An important finding of the Gulasekaran (2004) study is that misleading inferences are more likely at low levels of temporal aggregation. Therefore, moving towards more disaggregated data
would not be of much help unless the observation frequency coincides with the causal lag. Looking for data with a frequency that coincides with the causal lag is a far-fetched goal. Moreover, for certain variables such as GDP, there is a limit to the increase in the frequency of observations beyond which meaningful time series aggregates do not exist. (See Hoover, 2001, Chapter 6 for an eloquent discussion on this.) On the other hand, temporal aggregation creates contemporaneous correlations even when such correlations are absent in the non-aggregate process and as temporal aggregation increases contemporaneous correlations may be all that is left between the series. As a result one may not find Granger-causality at all. Making causal inferences from contemporaneous correlations is a challenging and much needed task.

Hoover (2001) has proposed a very promising methodology for causal inference based on an intervention analysis. Using the logic that interventions such as strikes, wars, and policy changes do not alter a true causal relationship, Hoover suggests examining conditional and marginal distributions to see whether they are affected by interventions. Hoover’s approach is closely related to testing for super exogeneity (see Hendry, 1995). The applications he has reported, however, rely on error correction formulations which are unfortunately subject to the distortions of temporal aggregation.

Swanson and Granger (1997) have used a graph-theoretic approach advocated by philosophers and computer scientists (see Pearl, 2000) to assign a causal ordering to contemporaneous links in a structural vector autoregression (SVAR) model. Demiralp and Hoover (2003) have examined the reliability of this approach and conclude that the graph-theoretic approach combined with non-sample information may lead to a more realistic causal ordering. This approach, the SVAR approach in general, faces a number
of difficulties under temporal aggregation. First, the results in Gulasekaran and Abeysinghe (2002) and Gulasekaran (2004) indicate that though unidirectional relations create contemporaneous correlations with an unambiguous sign, an ambiguity may arise if the non-aggregate process is a feedback relation with a positive and negative feedback loop. Depending on the parameter configuration of the non-aggregate process, the contemporaneous correlation of the aggregate process may take positive, negative or zero values from which it would be difficult to infer the actual feedback relation that exists (see also Ericsson et al., 2001). Second, since causal ordering is performed on the basis of residuals of an unrestricted VAR there is a possibility that the lagged variables in the unrestricted VAR acting as proxies for the omitted contemporaneous relations thereby leading to distorted residual correlations. Therefore, the final causal ordering may have to be done by estimating the full SVAR model. Third, even if the causal ordering of contemporaneous links is done successfully, temporal aggregation may distort the dynamics of the VAR model in such a way that the impulse responses may become misleading (Granger, 1988; Granger and Swanson, 1992; Marcellino 1999).

Another promising approach that has attracted many researchers in recent years is continuous time modeling (see Bergstrom, 1990; Phillips, 1991; Bergstrom et al., 1994; Chambers, 1999 and the references therein). Although many microeconomic processes such as output from a production line are discrete, at an economy-wide macro level a continuous process may provide the best approximation to the underlying discrete process. Unfortunately, the level of technical difficulty involved in deriving the exact

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1 For example, since early 1980s (until the onset of the Asian financial crisis) the Monetary Authority of Singapore has used exchange rate management as a means of controlling inflation. Higher imported inflation prompts for an appreciating exchange rate policy that in turn lowers the inflation rate. Therefore, the two variables form a positive and negative feedback loop. However, the observed data hardly show this relationship.
discrete representation that corresponds to the observed data of a complex continuous process and efficient estimation of the parameters of the continuous process have constrained the empirical work in this area to a greater extent.

An alternative route to take is to focus on relationships that are invariant to temporal aggregation. It is well established now that cointegrating relations remain invariant to temporal aggregation (Stock, 1987; Granger, 1990; Phillips, 1991; Pierse and Snell, 1995; Granger and Siklos, 1995; Franses and Boswijk; 1996; Marcellino, 1999). They are also not affected by linear seasonal filters (see Hendry, 1995, Section 15.6 and the references therein). Cointegration also implies G-causality (Granger, 1986) though the direction is uncertain. Establishing the direction of causality from a cointegrating relation is an important research agenda and some apparatus under weak exogeneity already exists for this purpose (Engle et al., 1983, Hendry 1995, Johansen 1995). As the level of temporal aggregation increases, a stationary VAR(\(p\)) process may tend towards VAR(0) by absorbing all causal information into contemporaneous links. However, a cointegrated VAR(\(p\)) process cannot shrink below VAR(1) because of the presence of unit roots.\(^2\) As a result some adjustment coefficients of the error correction model have to remain non-zero regardless of the level of temporal aggregation. Therefore, weak exogeneity under cointegration and Granger causality are closely linked concepts.

Since it is of critical importance to make causal inference from relations that are invariant to temporal aggregation and given the impracticality of searching for the non-aggregate forms of the data series we propose a methodology to make causal inference from temporally aggregated data series that are cointegrated. As we shall see in Section

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\(^2\) If \(\lambda_1, \lambda_2, ..., \lambda_n\) are the roots of the non-aggregate autoregressive process then \(\lambda_1^m, \lambda_2^m, ..., \lambda_n^m\) are the roots of the aggregated process, where \(m\) is the order of aggregation (Marcellino, 1999).
2, our focus is to make causal inference from the adjustment coefficients of an error correction model that will help in establishing weak exogeneity and contemporaneous conditioning in a vector model.

We present the analytical results in Sections 2 and 3 and summarize our operational solution, the sign rule, in Section 4. We also carried out a number of Monte Carlo simulations and these results are summarized in Section 5. We then present three applications in Section 6 to illustrate how the sign rule works. The concluding section highlights the key features of the exercise.

2. Analytical tools

Consider the vector error correction model (VECM):

$$
\Delta y_t = \alpha \beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + \Gamma_2 \Delta y_{t-2} + \ldots + \Gamma_{p-1} \Delta y_{t-(p-1)} + \phi D_t + e_t
$$

(1)

where $y_t$ is an $(n \times 1)$ vector of I(1) variables, $D_t$ contains deterministic terms such as the constant and time trend and $u_t = \beta' y_t$ is an $(r \times 1)$ vector of cointegrating relations (Johansen, 1995). Note that $\text{Var}(e_t) = \Sigma$ is not a diagonal matrix in general. If we can impose a meaningful causal ordering on the contemporaneous relations such that $B_0 e_t = \varepsilon_t$ and $\text{Var}(\varepsilon_t) = \Omega$ is diagonal, where $\varepsilon_t$ are the fundamental innovations, then we can formulate a structural VECM by pre-multiplying (1) by $B_0$. When one is uncertain about the causal direction between two variables, imposing restrictions on $B_0$ may become arbitrary. This problem can be resolved by making a data-based causal inference. The problem is, however, that temporal aggregation alters
\(\alpha\) s and \(\Gamma\) s and if this happens in a distortionary way meaningful causal inference may not result.

Since our focus in this paper is on \(\alpha\) we assume that the data series are sufficiently temporally aggregated such that \(\Gamma\) s in the aggregated process are practically zero and we work with the model:\(^3\)

\[
\Delta y_t = \alpha \beta y_{t-1} + e_t. \tag{2}
\]

For the convenience of subsequent derivations we have dropped the deterministic term from (2). Although the methodology proposed here is quite amenable to higher order VAR models with high-frequency data, our advice is to refrain from making causal inference from the \(\Gamma\) s.

As stated earlier, cointegration (also unit roots) is invariant to temporal aggregation. This is in general true for static relations. Note also that if \(y\) consists of both flow and stock variables we have to apply the same transformation, temporal aggregation or averaging, to all the variables to keep \(\beta\) invariant. Therefore, it is advisable to use temporal averaging of all the variables when both flow and stock variables enter the same model. Since \(\beta\) is invariant to temporal aggregation we proceed with the assumption that \(\beta\) is known and concentrate on \(\alpha\).

If the \(n\) variables are partitioned into two groups such that \(y_{1t}\) is \((n_1 \times 1)\) and \(y_{2t}\) is \((n_2 \times 1)\), \(n_1 + n_2 = n\) with \(\alpha\) and \(\beta\) split conformably then the ECM can be written as

\[
\begin{pmatrix}
\Delta y_{1t} \\
\Delta y_{2t}
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix} \begin{pmatrix}
\beta_{11}' & \beta_{12}'
\end{pmatrix} \begin{pmatrix}
y_{1t-1} \\
y_{2t-1}
\end{pmatrix} + \begin{pmatrix}
e_{1t} \\
e_{2t}
\end{pmatrix}. \tag{3}
\]

\(^3\) Johansen (1995) argues, based on experience, that seasonally adjusted quarterly data are often well modeled as VAR(2). If longer lags are required he suggests to look for omitted variables and increase \(n\) instead of \(p\). For example, consider that the \(n\) variables are well modeled as a VAR(1) process. If we throw away half of the variables then the remaining half requires a VARMA(2,1) model instead of a VAR(1).
Johansen (1995, Ch. 8) shows that if $\alpha_2 = 0$ then $y_{2t}$ is weakly exogenous for $\beta$ and $\alpha_1$. Writing (3) in levels it can be seen that $\alpha_2 = 0$ (assuming $\beta \neq 0$) also implies the presence of Granger causality from $y_{2t}$ to $y_{1t}$ though we cannot exclude the possibility of lagged feedback effects since they may have disappeared into contemporaneous links (see also Mosconi and Giannini, 1992).

Following Hendry and Mizon (1998) Granger causality associated with the adjustment coefficients may be referred to as causality in levels and that associated with short-run coefficients as causality in differences. Although the former implies the latter, the converse does not necessarily hold. Since causality in levels bears on weak exogeneity that forms the basis for super exogeneity which is essential for policy evaluations, the causality in levels plays an important role in practice. Since policy variables in general are subject to feedback effects, causality in differences does not play such a useful role in policy evaluations.

2.1 The sign of the adjustment coefficient

Johansen (1995) draws attention to “correct” and “wrong” signs of the adjustment coefficients in a number of places. In this section we define the “correct sign” formally for $r = 1$ and discuss the consequences of a “wrong sign”.

When $r = 1$, the $i$th equation of (2) can be written as

$$y_{it} = y_{i,t-1} + \alpha_i u_{i,t-1} + e_{it}, \quad (4)$$

where $u_t = \sum_{i=1}^{n} \beta_{i} y_{i,t}$.

The long run equilibrium implies that $u_{t-1} = 0$ which gives:
\[ y_{it-1} = \frac{-1}{\beta_i} (\beta_1 y_{it-1} + \beta_2 y_{2t-1} + \ldots + \beta_{i-1} y_{i-1} + \beta_{i+1} y_{i+1} + \ldots + \beta_n y_{nt-1}) . \]

If the system is in disequilibrium at date \( t - 1 \) then either \( u_{t-1} > 0 \) or \( u_{t-1} < 0 \).

Case 1: \( \beta_i > 0 \)

If \( u_{t-1} > 0 \), then
\[ y_{it-1} > \frac{-1}{\beta_i} (\beta_1 y_{it-1} + \beta_2 y_{2t-1} + \ldots + \beta_{i-1} y_{i-1} + \beta_{i+1} y_{i+1} + \ldots + \beta_n y_{nt-1}) \]
and we expect \( \alpha_i u_{t-1} < 0 \) in (4) in order to achieve the equilibrium. Since \( u_{t-1} > 0 \), \( \alpha_i \) has to be negative (\( \alpha_i < 0 \)).

If \( u_{t-1} < 0 \), then
\[ y_{it-1} < \frac{-1}{\beta_i} (\beta_1 y_{it-1} + \beta_2 y_{2t-1} + \ldots + \beta_{i-1} y_{i-1} + \beta_{i+1} y_{i+1} + \ldots + \beta_n y_{nt-1}) \]
and we expect \( \alpha_i u_{t-1} > 0 \) in (4) in order to achieve the equilibrium. Since \( u_{t-1} < 0 \), \( \alpha_i \) has to be negative (\( \alpha_i < 0 \)).

Thus, if \( \beta_i > 0 \), then \( \alpha_i < 0 \) regardless of the sign of the disequilibrium term \( u_{t-1} \).

Case 2: \( \beta_i < 0 \)

If \( u_{t-1} > 0 \), then
\[ y_{it-1} > \frac{-1}{\beta_i} (\beta_1 y_{it-1} + \beta_2 y_{2t-1} + \ldots + \beta_{i-1} y_{i-1} + \beta_{i+1} y_{i+1} + \ldots + \beta_n y_{nt-1}) \]
and we expect \( \alpha_i u_{t-1} > 0 \) in (4) in order to achieve the equilibrium. Since \( u_{t-1} > 0 \), \( \alpha_i \) has to be positive (\( \alpha_i > 0 \)).
If $u_{t-1} < 0$, then $y_{t-1} > -\frac{1}{\beta_i} (\beta_1 y_{t-1} + \beta_2 y_{2t-1} + ... + \beta_{i-1} y_{(i-1)t-1} + \beta_{i+1} y_{(i+1)t-1} + ... + \beta_n y_{nt-1})$

and we expect $\alpha_i u_{t-1} < 0$ in (4) in order to achieve the equilibrium. Since $u_{t-1} < 0$, $\alpha_i$ has to be positive ($\alpha_i > 0$).

Thus, if $\beta_i < 0$, then $\alpha_i > 0$ regardless of the sign of the disequilibrium term, $u_{t-1}$.

We can, therefore, see that the long-run equilibrium holds if the sign of $\alpha_i$ is the opposite of that of $\beta_i$. We define this as the “correct sign” of the adjustment coefficient when $r = 1$. Since the $\beta$ vector is invariant to temporal aggregation we can determine the expected sign of $\alpha_i$ from that of $\beta_i$ under any level of temporal aggregation. We will call this the “sign rule”.

The problem, however, is that the long-run equilibrium may hold even if the sign is wrong. To see this, from the $n$ equations in (4) obtain $u_t = \rho u_{t-1} + \varepsilon_t$, where $\rho = 1 + \alpha' \beta$, $\alpha = (\alpha_1, ..., \alpha_n)'$, $\beta = (\beta_1, ..., \beta_n)'$ and $\varepsilon_t$ is white noise. Co-integration in this I(1) framework requires $|\rho| < 1$, which implies $-2 < \alpha' \beta < 0$. Note that $\rho$ measures the degree of co-integration: $|\rho| \to 0$ implies a higher degree of co-integration and $|\rho| \to 1$ implies a lower degree. Now consider that the variables in the model are arranged such that the first $n_1$ $\beta$ coefficients are positive and the second $n_2$ $\beta$ coefficients are negative.

Let $\beta_1$ and $\beta_2$ represent these vectors and let the corresponding $\alpha$ vectors be $\alpha_1$ and $\alpha_2$ respectively. Assume that $\alpha_1$ is correctly signed with a negative sign and $\alpha_2$ is wrongly signed with a negative sign too. Given the inequality $-2 < \alpha_1' \beta_1 + \alpha_2' \beta_2 < 0$, if $|\alpha_1' \beta_1| > |\alpha_2' \beta_2|$ we get $|\rho| < 1$ even with the wrong sign. In other words, if the adjustment
towards equilibrium is dominated by the adjustment coefficients with the correct sign co-integration continues to hold. However, the wrong sign lowers the degree of co-integration (increases the absolute value of $\rho$). How frequent the co-integration with the wrong sign is an empirical question. As we shall see later the wrong sign appears to result primarily from temporal aggregation.

2.2 One cointegrating vector

In the following analysis we confine to the case where $r = 1$. The analytics for the case where $r > 1$ remains to be worked out. The case of $r = 1$ is by no means too restrictive in practice. Large econometric models are usually presented with a set of parsimonious equations each involving a smaller number of variables which tend to be cointegrated with one cointegrating vector. These variables may be thrown into the VAR framework if one is interested in causality analysis. It is also well known that VAR models do not perform well when the dimension is too large.

Since our focus is on the relationship between the $i$th variable and the $u$ term under the assumption of known $\beta$ ($\beta$ can be estimated super-consistently), we concentrate on a bivariate case to derive the analytical results, but these results apply in general to multivariate models. We assume that the non-aggregate process is in the form of (3) with the resulting two equations written as

\[ \Delta y_{1t} = \alpha_1 u_{t-1} + e_{1t}, \]  
\[ \Delta y_{2t} = \alpha_2 u_{t-1} + e_{2t}, \]  
\[ \text{(5')} \]

where $u_t = \beta_1 y_{1t} + \beta_2 y_{2t}$. We assume that
\[
\begin{bmatrix}
  e_{1t} \\
  e_{2t}
\end{bmatrix}
\sim iidN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right).
\]

The zero contemporaneous covariance between the two error terms underlies the assumption that the observation frequency coincides with the causal lag. As before \(u_t = \rho u_{t-1} + \epsilon_t\), where \(\rho = 1 + \alpha_1 \beta_1 + \alpha_2 \beta_2\) and \(\epsilon_t = \beta_1 e_{1t} + \beta_2 e_{2t}\) with zero mean and variance \(\sigma^2_\epsilon = \beta_1^2 \sigma^2_1 + \beta_2^2 \sigma^2_2\).

Let \(w_{1t} = \Delta y_{1t}\) and \(w_{2t} = \Delta y_{2t}\). For \(i=1,2\), the variances and covariances of the non-aggregate process in (5) and (5') can be written as:

\[
\gamma_u(k) = E(u,u_{t-k}) = E(u,u_{t+k}) = \rho^k \sigma^2_\epsilon / (1 - \rho^2) \quad \forall k
\]

\[
E(e_u u_{t+k}) = \begin{cases} 
0 & \text{if } k < 0 \\
\rho^k \beta_i \sigma^2_\epsilon & \text{if } k \geq 0
\end{cases}
\]

\[
\gamma_w(k) = E(w_{it}, w_{it-k}) = \begin{cases} 
\alpha^2_i \gamma_u(k) + \sigma^2_i & \text{if } k = 0 \\
\alpha^2_i \gamma_u(k) + \alpha_i \rho^{k-1} \beta_i \sigma^2_\epsilon & \text{if } k > 0
\end{cases}
\]

\[
\gamma_w(-k) = \gamma_w(k)
\]

\[
\gamma_{uw}(k) = E(w_{it}, u_{t+k}) = \begin{cases} 
\alpha_i \frac{\rho^{k-1}}{1-\rho} \sigma^2_\epsilon & \forall k > 0 \\
\alpha_i \frac{\rho^{k-1}}{1-\rho} \sigma^2_\epsilon + \rho^{k-1} \beta_i \sigma^2_\epsilon & \forall k \leq 0.
\end{cases}
\]

Let \(y_{1t}\) and \(y_{2t}\) (\(t=1,2,\ldots,N\), \(T = mN\)) be the \(m\)-period non-overlapping aggregates of \(y_{1t}\) and \(y_{2t}\) respectively and let \(W_{1t} = \Delta y_{1t}\) and \(W_{2t} = \Delta y_{2t}\). We now consider estimating the following aggregated process:

\[
W_{1t} = \alpha^*_1 U_{t-1} + E_{1t}
\]

\[
W_{2t} = \alpha^*_2 U_{t-1} + E_{2t}
\]
where \( Y_{it} = \sum_{j=mt}^{mr} y_{ij} \), \( U_{it} = \sum_{j=mt}^{m(t-1)} u_{ij} \) and \( E_{it} \) the corresponding error process.\(^4\)

The OLS estimates \( \hat{\alpha}_i \), \( p \lim \hat{\alpha}_i \) and the \( t \) statistics are given by:

\[
\hat{\alpha}_i^* = \frac{\sum W_i U_{t-1}}{\sum U_{t-1}^2}, \quad p \lim \hat{\alpha}_i^* = \frac{Y_{iU}(1)}{Y_{U}(0)}
\]

\[
t_i^* = \frac{\hat{\alpha}_i^*}{[\text{var}(\hat{\alpha}_i^*)]^{1/2}}
\]

where

\[
\text{var}(\hat{\alpha}_i^*) = \frac{\sigma_{E_i}^2}{\sum U_{t-1}^2} = \frac{\sigma_{E_i}^2}{N Y_{U}(0)}, \quad \sigma_{E_i}^2 = \text{var}(E_i) = \gamma_{ii}^W(0) + \hat{\alpha}_i^* \gamma_{iU}(0) - 2\hat{\alpha}_i^* Y_{iU}(1).
\]

Using Proposition A.1 given in the Appendix that establishes the relationship between covariances of the aggregated and the non-aggregate processes we get the following relations, again for \( i=1,2 \):

\[
\gamma_{U}(k) = (1 + L + \ldots + L^{m-1})^2 \gamma_u (mk + (m-1)) \quad \forall k.
\]

\[
\gamma_{iU}(k) = (1 + L + \ldots + L^{m-1})^3 \gamma_{iu} (mk + (m-1)) \quad \forall k
\]

\[
\gamma_{ii}^W(k) = (1 + L + \ldots + L^{m-1})^4 \gamma_{ii}^W (mk + 2(m-1)).
\]

These expressions provide the link between the parameter estimates and the \( t \)-statistics of the aggregated process and the parameters of the non-aggregate process in order to derive a quantitative evaluation of the impact of temporal aggregation.

\(^4\) In addition to temporal aggregation we also examined systematic sampling. In general, systematic sampling does not lead to serious distortions in the adjustment coefficients. In the interest of space we present the results for temporal aggregation only.
3. Distortions

There are three cases of interest with regard to Granger causality in the non-aggregate process: (i) no causality, (ii) unidirectional causality and (iii) mutual causality. The first case clears through without a problem. If the two series are not related in the non-aggregate process then \( \alpha_i = 0, \ i = 1,2, \) and from (9), \( \gamma_{iu}(k) = 0, \ \forall k > 0. \) Further, from (15) \( \gamma_{iu}(k) = 0, \ \forall k > 0 \) and from (11) \( \hat{\alpha}_i^* = 0. \) Thus, if there is no Granger causality between the series in the non-aggregate process then there will be no Granger causality between them in the aggregated process. In fact, this is valid for the short run dynamics as well (Gulasekaran, 2004).

3.1 Unidirectional causality in the non-aggregate process

To evaluate this case we set \( \alpha_2 = 0 \) so that Granger causality runs from \( y_{2t} \) to \( y_{1t}. \) For the aggregate process from (11) we get \( p\lim \hat{\alpha}_2^* = \gamma_{2u}(1)/\gamma_{2u}(0). \) To evaluate this expression first note from (11) for the non-aggregate process when \( \alpha_2 = 0 \) we get

\[
\gamma_{2u}(k) = \begin{cases} 
\alpha_2 \frac{\rho^{k+1}}{1-\rho} \sigma^2_e = 0 & \forall k > 0 \\
\alpha_2 \frac{\rho^{-k+1}}{1-\rho} \sigma^2_e + \rho^{-k} \beta_2 \sigma^2_2 = \rho^{-k} \beta_2 \sigma^2_2 & \forall k \leq 0.
\end{cases}
\]  

(17)

Now using the expressions in (14) to (16), the numerator of \( p\lim \hat{\alpha}_2^* \) can be written in terms of the covariances and the parameters of the non-aggregate process as
\[ \gamma_{2U}(1) = \left(1 + L + L^2 + \ldots + L^{m-1}\right) \gamma_{2u}(2m-1) \]
\[ = \left(c_0 + c_1L + \ldots + c_{3(m-1)}L^{3(m-1)}\right) \gamma_{2u}(2m-1) \]
\[ = c_0 \gamma_{2u}(2m-1) + c_1 \gamma_{2u}(2m-2) + \ldots + c_{2m-1} \gamma_{2u}(0) + c_{2m} \gamma_{2u}(-1) + \ldots + c_{3(m-1)} \gamma_{2u}(-(m-2)) \]
\[ = c_{2m-1} \gamma_{2u}(0) + c_{2m} \gamma_{2u}(-1) + \ldots + c_{3(m-1)} \gamma_{2u}(-(m-2)) \]
\[ = \beta_2 \sigma_2^2 \left(c_{2m-1} + c_{2m} \rho + c_{2m+1} \rho^2 + \ldots + c_{3(m-1)} \rho^{m-2}\right) \]

(18)

where \( c_j \) are positive constants such that \( c_{2m-1} > c_{2m} > \ldots > c_{3(m-1)} \). By plugging (17) in (18) we can see that \( \gamma_{2U}(1) \neq 0 \), hence \( p \lim \hat{\alpha}_2^* \neq 0 \). What is most important to note here is that the sign of \( p \lim \hat{\alpha}_2^* \) would be the same as that of \( \beta_2 \). This is obvious from (18) when \( \rho \geq 0 \). For \( \rho < 0 \) we can re-write the last expression of (18) as:

if \( m \) is even,
\[ \gamma_{2U}(1) = \beta_2 \sigma_2^2 \left[(c_{2m-1} + c_{2m} \rho) + (c_{2m+1} + c_{2m+2} \rho) \rho^2 + \ldots + \right. \left(c_{3(m-1)-2} + c_{3(m-1)-1} \rho) \rho^{m-4} + c_{3(m-1)} \rho^{m-2}\right] \]

and if \( m \) is odd,
\[ \gamma_{2U}(1) = \beta_2 \sigma_2^2 \left[(c_{2m-1} + c_{2m} \rho) + (c_{2m+1} + c_{2m+2} \rho) \rho^2 + \ldots + \right. \left(c_{3(m-1)-1} + c_{3(m-1)} \rho) \rho^{m-3}\right]. \]

Since \( c_i + \rho c_{i+1} > 0, \ i = 2m - 1, 2m, \ldots, 3m - 1 \), it is now easily seen that the sign of \( \gamma_{2U}(1) \), hence the sign of \( p \lim \hat{\alpha}_2^* \), would be the same as that of \( \beta_2 \) regardless of the sign of \( \rho \). Recall now that the sign rule requires the adjustment coefficient and the cointegrating coefficient to take opposite signs. This predictable sign distortion we see in the above result is the key guidance for us in the assessment of the causal direction in the non-aggregate process.
Although the above result clearly establishes that temporal aggregation creates a causality distortion, it is worthwhile investigating how important these distortions are for practical purposes. For this we use the above analytical expressions to derive some numerical results using the following setting. We use the normalized co-integrating vector \((1, \beta_2)\) and consider \(m=3\) and \(m=12\) to represent aggregating monthly data to quarterly and annual figures.\(^5\) To assess the impact of the degree of co-integration we consider values of \(\rho\) in the range \(-0.95\) to \(0.95\). This is the same as setting \(\alpha_i(=\rho-1)\) within the range \(-1.95\) to \(-0.05\). We also vary the values of \(\beta_2\) within the range \(-20\) to \(20\) to see whether the magnitude of \(\beta_2\) plays any role in creating the distortions. For the computation of the \(t\) statistics we consider three combinations of \(m\) and \(N\). The numerical results are given in Tables 1 to 4.\(^6\)

Co-integration implies that at least one of the adjustment coefficients has to be non-zero. As expected, \(\hat{\alpha}_1^*\) remains negative and highly statistically significant regardless of the level of aggregation and the sample size (Tables 1 and 2). The magnitude of \(\beta_2\) seems to matter when \(\alpha_i\) is very small.

The interesting case is \(\hat{\alpha}_2^*\) which takes a non-zero value with a predictable sign-distortion. Table 3 shows that the limiting values of \(\hat{\alpha}_2^*\) are small in magnitude for certain cases especially when \(m=3\). The magnitude of both \(\alpha_1\) and \(\beta_2\) play a role in the creation of a non-zero \(\hat{\alpha}_2^*\). Nevertheless Table 4 shows that the impact of the magnitude of \(\beta_2\)

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\(^5\) Since monthly data are the aggregates of daily or hourly data we have to set \(m\) to very large values which render our analytical expressions unmanageable.

\(^6\) These sample sizes are chosen to be compatible with the Monte Carlo experiments in Lahiri and Mamingi (1995), Choi and Chung (1995) and Mamingi (1996). Since we compute \(t\) statistics using the limiting values of the parameter estimates we conducted a limited number of Monte Carlo experiments (with 10,000 replications) to assess the validity of our theoretical results for the sample sizes considered. The Monte Carlo results are the same as our theoretical results.
disappears from the $t$ statistics. However, when $\alpha_1$ is close to -1 (high degree of cointegration) the $t$ statistics are highly significant regardless of the level of aggregation and the sample size. An increase in $m$ or $N$ renders more statistically significant $t$ statistics. These results concur with the Monte Carlo results in Mamingi (1996).

_____________________

Tables 1-4

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3.2 Mutual causality in the non-aggregate process

If cause and effect are viewed as asymmetrical concepts, as is done in many philosophical traditions, then the question of mutual causality does not arise. However, since this exercise is not only about causality, it is also about contemporaneous conditioning in a multivariate model under the assumption of weak exogeneity, it is useful to examine how the sign distortion shown in the previous section holds under mutual causality in levels.

In this case both $\alpha_1$ and $\alpha_2$ take non-zero values. Given $\alpha_1$, when $\alpha_2$ reaches zero we can see from (17) and (18) a repeat of the results in the previous section, $p \lim \tilde{\alpha}_2^*$ takes a non-zero value but with a sign distortion; it takes the same sign as that of $\beta_2$. Again to obtain a numerical assessment we followed the procedure of the previous section. In this case, however, since $\alpha_2 = (\rho - 1 - \alpha_1) / \beta_2$, the computational setting is a lot more involved. We computed a large number of tables using various combinations of the parameter values. To conserve space we provide only a summary here.
In all computations we fixed $\alpha_1$ to the range -0.95 to -0.05. A representative case for $\hat{\alpha}_2^*$ is given in Table 5 which presents the $t$ statistics for $\hat{\alpha}_2^*$ under one set of parameter configuration. In this table $\alpha_2$ is positive and varies from cell to cell but takes on smaller values towards the top-left corner. In general we observe the following results. 1. When both $\alpha_1$ and $\alpha_2$ are large in magnitude distortions do not occur. 2. When they are both small in magnitude the one with the smaller magnitude may become statistically insignificant. 3. When one of them is very small relative to the other one, the smaller one may take on the wrong sign and become statistically significant (see, top left-corner of Table 5). This effect magnifies when $m$ increases.

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Table 5

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4. How to test for causality in levels with temporally aggregated data?

Although temporal aggregation tends to distort the adjustment coefficients, the sign rule established in Section 2 and the results in Section 3 show that we are still be able to reach the correct conclusion about the causal direction based on the sign of the adjustment coefficients. To repeat the sign rule, if $\beta_i > 0$, $\alpha_i < 0$ and if $\beta_i < 0$, $\alpha_i > 0$.

The results on unidirectional causality are clear-cut. The non-zero adjustment coefficient remains highly significant regardless of the level of temporal aggregation and carries the correct sign. An adjustment coefficient with the wrong sign clearly indicates a causal distortion of the underlying zero coefficient.
However, if mutual causation in levels is allowed, then the inference is not that straightforward. Our results nevertheless show that when $\alpha_1$ is reasonably large $\hat{\alpha}_1^*$ always carries the correct sign and remains statistically significant. Therefore, a proper normalization (a selection of the dependent variable) with the help of non-sample information should make the inference easier. A weak mutual causal effect given by a small $\alpha_2$ then may manifest as a unidirectional relation in the aggregate process with a wrong sign, an indicator of causal distortion. However, we noted in Section 2.1 that cointegration is possible even with a wrong sign on $\alpha_2$ provided that its magnitude is smaller than that of $\alpha_1$. Although an observed wrong sign on $\hat{\alpha}_2^*$ may be purely due to temporal aggregation, it would be difficult to rule out an $\alpha_2$ with a wrong sign without non-sample information. As we shall see in our applications, it would not be difficult to find non-sample information to solve this identification problem. We, therefore, formulate the following rule for causal inference and contemporaneous conditioning.

First, determine the expected sign of the adjustment coefficients from the estimated cointegrating vector. If the estimated adjustment coefficient appears with the correct sign and is statistically significant then it reflects the underlying causal direction in the non-aggregate form. If the coefficient appears with the wrong sign then a causal distortion may have occurred and if such a conclusion is supported by non-sample information then we may treat it as resulting from a zero or near zero coefficient in the non-aggregate form.
5. Some Monte Carlo results

We carried out a limited Monte Carlo exercise based on bivariate and trivariate models by setting \( p=2 \) in (1), \( \alpha_1 \neq 0, \alpha_2 = \alpha_3 = 0 \) and \( \gamma_{1,j} \neq 0 \) to see how the sign rule works when short-run feedback effects (causality in differences) are present. The results were very promising, the sign distortion on \( \hat{\alpha}_2^* \) and \( \hat{\alpha}_3^* \) was almost 100% for \( m=3 \) and 100% for \( m=12 \) regardless of the presence of short-run feedback effects. These results can be obtained from the authors.

6. Applications

6.1 Exchange rate, direct vs. cross

The relationship between the three major exchange rates, US$, Deutsch Mark (DM) and Japanese Yen, provides a good illustration of how the sign rule works under unidirectional causality in a multivariate setting. Theoretically the direct Yen/DM rate should be the same as the cross rate derived from US$/DM and US$/Yen rates. Any deviations will open up arbitrage opportunities for profiteering. However, some deviations may still be observed when transaction costs are higher than the potential profits. Therefore, \( \log(\text{Yen}/\text{DM}) - \log(\text{US}$/\text{DM}) + \log(\text{US}$/\text{Yen}) \) forms a co-integrating relation with the co-integrating vector \((1, -1, 1)\). Figure 1 shows the deviations of the daily direct rate from the cross rate. As can be expected these deviations are very small and center around zero. Somewhat surprisingly, though, they show some heteroscedastic behavior. Both AIC and SBC pick a VAR(1) for the three rates (Yen/DM, US$/DM, US$/Yen) all in logarithms. The residual correlation matrix is not diagonal that reflects the systematic sampling of the daily rates. Although the residuals are free from serial
correlation, both normality and heteroscedasticity tests fail. We ignore this and proceed to estimate the adjustment coefficients by imposing the above co-integrating vector.\footnote{The estimated cointegrating vector virtually coincide with (1, -1, 1) vector under both systematic sampling and temporal aggregation.} The results (based on the Johansen ML method) are reported in Table 6. The results under both daily rates and systematically sampled weekly rates show that only $\alpha_1$ is non-zero. This is also what we expect apriori. However, under temporally averaged weekly rates both $\alpha_1$ and $\alpha_3$ turn out to be non-zero. Nevertheless, the wrong sign of $\alpha_3$ provides the warning sign. Combined with non-sample information that only $\alpha_1$ could be non-zero we could safely conclude that non-zero $\alpha_3$ is a result of temporal averaging. Constraining both $\alpha_2$ and $\alpha_3$ to zero brings the estimate of $\alpha_1$ closer to unity.

6.2 Stock market and car quota premium in Singapore

This is an interesting example because one variable is available in non-aggregate form. To curb the car population, the Singapore government implemented a car quota system in August 1990. To buy a new car the buyers first have to buy a piece of paper called the certificate of entitlement. The price of this paper, known as the quota premium (QP), is decided through a monthly bidding process. The monthly data of QP are not contaminated by any form of aggregation or systematic sampling.

A key determinant of QP of luxury cars is the performance of the stock market, captured by the stock price index compiled by the Stock Exchange of Singapore (Lai,
Monthly data over 1990M8-1999M4 show that these two variables (in logarithms) are cointegrated and their relationship is well represented by a VAR(1) process with causality running from stock price to QP. Stock prices (in log) follow a random walk. Moreover, the two error processes are also uncorrelated ($\sigma_{12} = 0$). We write the ECM formulation as:

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix},$$  \hspace{1cm} (19)

where $y = \ln(QP)$ and $x = \ln(\text{Stock price index})$, $\alpha_2 = 0$. Because of the short data span we temporally averaged data up to six months. The results for $m=1$ (no aggregation) through $m=6$ are given in Table 7. The results show that $\hat{\beta}_2$ remains roughly the same as $m$ increases. Being a cointegrated VAR(1) with $\sigma_{12} = 0$ temporal aggregation does not create contemporaneous correlation between the residual processes ($r_{12}$ remains close to zero). However, the magnitude of $\hat{\alpha}_1^*$ increases steadily and remains highly significant. The magnitude of $\hat{\alpha}_2^*$ also tends to increase though not steadily and becomes statistically significant at the 10% level when $m=4$ and $m=6$. If one had only the temporally averaged data (say biannual) the wrong (negative) sign of $\hat{\alpha}_2^*$ provides the warning signal. Combined with the information that (log) stock prices follows a random walk one could safely conclude in this case that causality is unidirectional from stock prices to QP.

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Table 7

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22
6.3 Tax revenue and government expenditure in the US

Barro’s (1979) tax smoothing hypothesis offers an interesting contrast to Granger causality testing. Under the assumption that spending causes taxes Barro’s model implies that the income tax rate follows a random walk. If the tax rate truly follows a random walk, the standard Granger-causality test will fail to establish the causal direction embodied in Barro’s model. Since taxes following a random walk could be consistent with some other hypotheses, many researchers have tested the tax smoothing hypothesis by testing the other implications of the model, see for example, Sahasakul (1986), Huang and Lin (1993) and Ghosh (1995). Hoover (2001), however, applied his intervention approach to test for the causal direction between taxes and spending and observed mixed results over different time periods. Hoover used quarterly data from 1947 to 1989 in his analysis. In this section we use annual data (1946-2002) and examine how cointegration and the sign rule shed light on the causal direction between taxes and spending.

The data series used in this section are the following. $T =$ real federal government receipts (nominal series deflated by the GNP deflator, $P$), $G =$ real federal government expenditure net of interest payments (nominal series deflated by $P$), $Y =$ real GNP, $y = \ln Y$, $\Delta y =$ GNP growth rate ($\%$), $\pi = (\Delta \ln P)100 =$ inflation rate, $\tau = (T/Y)100 =$ income tax rate and $g = (G/Y)100 =$ spending rate. Data plot and ADF tests support the assumption that $\tau$, $g$, and $\pi$ are I(1) processes.

The literature usually focuses on the budget surplus as a ratio of GNP, $\tau - g$. Figure 2 plots this data series. Although the tax smoothing hypothesis predicts $\tau - g$ to be a stationary series (see Huang and Lin, 1993; Ghosh, 1995) the plot in Figure 2 casts

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8 The data series were taken from the same source that Hoover (2001) used, National Income and Product Accounts as reported in CITIBASE.
doubts on the stationarity of the series. An ADF regression with two lags of $\Delta (\tau - g)_t$ produces a t statistic of -2.62 which is insignificant at the 5% critical value of -2.92. In fact the $\tau - g$ series suggests some level shifts: on average a budget surplus in the period 1946-1970, a deficit in the period 1971-1993 and a large surplus in the period 1994-2002.

Although the budget surplus ($\tau - g$) may not necessarily be stationary, $\tau$ and $g$ form a strong cointegrating relationship with a different cointegrating vector. An OLS regression of $\tau$ on $g$ produces highly stable recursive parameter estimates with some small departures occurring after 1994. Based on average recursive OLS estimates we obtain the cointegrating relation $z_t = \tau_t - 0.25g_t - 13$. This series is plotted in Figure 3.

An ADF regression based on one lag of $\Delta z_t$ produces a t = -4.155 which is significant at the 1% critical value of -3.552.

We use the following specification to examine the adjustment coefficients $\alpha_1$ and $\alpha_2$, $\alpha_1$ is expected to be negative and $\alpha_2$ positive. Following Hoover (2001) we use $\Delta y_t$ and $\Delta \pi_t$ to remove non-policy effects from $\Delta \tau$ and $\Delta g$.\(^9\)

\[
\begin{align*}
\Delta \tau_t &= \delta_0 + \delta_1 \Delta \tau_{t-1} + \delta_2 \Delta g_{t-1} + \delta_3 \Delta y_t + \delta_4 \Delta \pi_t + \alpha_1 z_{t-1} + \varepsilon_{1t} \tag{20a} \\
\Delta g_t &= \lambda_0 + \lambda_1 \Delta \tau_{t-1} + \lambda_2 \Delta g_{t-1} + \lambda_3 \Delta y_t + \alpha_2 z_{t-1} + \varepsilon_{2t} \tag{20b}
\end{align*}
\]

FIML estimation of (20a) and (20b) produces results very similar to OLS estimates because the two error processes are empirically uncorrelated. We, therefore, proceed with OLS estimation. Figures 4 and 5 plot the recursive estimates from the two regressions. Figure 4 shows that the parameter estimates of the tax equation become unstable after

\(^9\) Hoover (2001) uses taxes and spending as a ratio of potential GNP to obtain the tax rate and the spending rate. He then regresses the tax rate on GNP-gap and inflation rate and the spending rate on the GNP-gap and uses the residuals from these regressions to study the causal direction. We also tried this approach but the measurement errors in potential GNP seem to cause parameter instabilities.
1994, the period of high budget surplus.\textsuperscript{10} Nevertheless, the adjustment coefficient estimate $\hat{\alpha}_1$ has the correct sign and is statistically significant. The parameter estimates of the spending equation shown in Figure 5 are more stable even during the high-budget-surplus period. But only the GNP growth rate and the constant term are statistically significant. After dropping the insignificant variables we obtain the following estimates over the period 1947-1994.\textsuperscript{11}

\[
\Delta \tau_i = -0.06 + 0.42 \Delta \tau_{i-1} + 0.16 \Delta g_{i-1} + 0.09 \Delta y_i + 0.12 \Delta \pi_i - 0.65 z_{i-1}
\]

\begin{center}
\begin{tabular}{cccccc}
          & (-0.37) & (3.51) & (2.28) & (2.44) & (2.77) & (-4.34) \\
$R^2$ & 0.71    & $\hat{\sigma}_1$ & 0.50 \\
\end{tabular}
\end{center}

\[
\Delta g_i = 0.52 - 0.14 \Delta y_i + 0.19 z_{i-1}
\]

\begin{center}
\begin{tabular}{ccc}
          & 2.37 & (-2.97) & 1.22 \\
$R^2$ & 0.32 & $\hat{\sigma}_2$ & 0.68 \\
\end{tabular}
\end{center}

The adjustment coefficient in (21a) has the correct sign and highly significant. The adjustment coefficient in (21b) is statistically insignificant. These results show that causality (in levels) runs from spending to taxes that concurs with Barro’s assumption. We have to note, however, that the recursive estimates of the adjustment coefficient in (21b), though statistically insignificant, are highly stable and bear the correct sign. Our previous results on the sign distortion indicate that a pure unidirectional relation produces an $\hat{\alpha}_2 < 0$ after temporal aggregation. Therefore, the correct sign of $\hat{\alpha}_2$ in (21b) seems to have resulted from a mild feedback system in the non-aggregate process. It should also be

\textsuperscript{10} The analysis after 1994 seems to require additional variables in the model.
\textsuperscript{11} The estimated equations pass the diagnostic tests available in PCGive except that the spending equation shows a mild heteroscedasticity. These results are not reported for brevity.
noted that (21a) clearly rejects a major implication of the tax smoothing hypothesis that
the tax rate (adjusted for the effect of $\Delta y_t$ and $\Delta \pi_t$) is a random walk.

7. Concluding remarks

Invariance of cointegrating relationships to temporal aggregation offers a promising path
for Granger causality testing. In this paper we propose using the error correction
formulation to infer the direction of causality between cointegrated variables. Temporal
aggregation distorts both the short-run coefficients and the adjustment coefficients in an
error correction model. Fortunately, unlike the short-run coefficients, the distortions on
the adjustment coefficients occur with a predictable sign-distortion. Based on these
findings we propose a sign-rule for making causal inferences from temporally aggregated
data.

Causal inference based on cointegration should be referred to as causality in
levels as opposed to causality in differences found in the short-run coefficients (Hendry
and Mizon, 1998). As we discussed in the text, causality found in the short-run
coefficients, regardless of the distortions due to temporal aggregation, is less useful for
policy analyses because policy variables in general show feedback effects. Causality in
levels, however, plays an important role in policy evaluations because of its connection to
super-exogeneity. The causal inference based on the adjustment coefficients also helps in
establishing weak exogeneity and contemporaneous conditioning of the VECM and
structural VAR models.
Appendix: The relationship between covariances of aggregate and non-aggregate processes

Temporal aggregation involves the construction of non-overlapping sums that can easily be obtained by defining the overlapping sum \((1+L+\ldots+L^{m-1})y_t\) and then systematically sampling this variable at every \(m\)th interval to obtain the aggregated variable \(Y_{\tau} = (1+L+\ldots+L^{m-1})y_{m\tau}\), \((\tau=1,2,\ldots,N; T=mN)\). Let \(w_t = (1-L)^d y_t\) and \(W_{\tau} = (1-L)^d Y_{\tau}\). The following result extends the univariate case considered by Stram and Wei (1986). (See Gulasekaran (2003) for further details.)

**Proposition A.1**

The covariance between the temporally aggregated series \(W_{i\tau}\) and \(W_{j\tau-k}\) can be expressed in terms of the covariance between the non-aggregate series \(w_i\) and \(w_{j-k}\) as, for \(k \geq 0\):

\[
\gamma^W_{ij}(k) = (1+L+L^2+\ldots+L^{m-1})^{d_i+d_j+2} \gamma^w_{ij}(mk+(d_j+1)(m-1))
\]

(A.1)

\[
\gamma^W_{ii}(k) = (1+L+L^2+\ldots+L^{m-1})^{2(d_i+1)} \gamma^w_{ii}(mk+(d_i+1)(m-1))
\]

(A.2)

where \(\gamma^w_{ij}(k) = \text{Cov}(W_{i\tau},W_{j\tau-k})\), \(\gamma^W_{ij}(k) = \text{Cov}(w_i,w_{j-k})\), \(L\) operates on the index of \(\gamma^w_{ij}(k)\) such that \(L\gamma^w_{ij}(k) = \gamma^w_{ij}(k-1)\), \(\gamma^W_{ij}(k) = \gamma^W_{ji}(-k)\) and \(d_i\) and \(d_j\) are integers representing orders of differencing applied to \(i\)th and \(j\)th series respectively.
Proof

Note that \((1 - L)Y_\tau = Y_{\tau} - Y_{\tau-1} = (1 + L + \ldots + L^{m-1})y_{m\tau} - (1 + L + \ldots + L^{m-1})y_{m(\tau-1)} = (1 - L^m) (1 + L + \ldots + L^{m-1})y_{m\tau}\). Therefore, \(W_{\tau} = (1 - L)^d Y_{\tau} = (1 - L^m)^d (1 + L + \ldots + L^{m-1})y_{m\tau}\) \(= (1 + L + \ldots + L^{m-1})^{d+1} w_{m\tau}\). Let \(F = L^{-1}\) be the lead operator such that \(Fz_{\tau} = z_{\tau+1}\) and let \(c_s\) and \(e_s\) be the coefficient of \(L^s\) of the polynomial \((1 + L + \ldots + L^{m-1})^{d+1}\) and \((1 + L + \ldots + L^{m-1})^{d+1}\) respectively.

\[
\gamma^w_{\gamma}(k) = E[W_{\tau}, W_{\tau-1}] \\
= E[(1 + L + \ldots + L^{m-1})^{d+1} w_{m\tau} (1 + L + \ldots + L^{m-1})^{d+1} w_{m(\tau-1)}] \\
= E[c_0 w_{m\tau} + c_1 w_{m\tau-1} + \ldots + c_{(d+1)\times(m-1)} w_{m\tau - (d+1)\times(m-1)}] \\
\quad + (e_0 w_{m\tau - mk} + e_1 w_{m\tau - m(k-1)} + \ldots + e_{(d+1)\times(m-1)} w_{m\tau - m(d+1)\times(m-1)}) \\
= c_0 [e_0 \gamma^w_{\gamma}(mk) + e_1 \gamma^w_{\gamma}(mk + 1) + \ldots + e_{(d+1)\times(m-1)} \gamma^w_{\gamma}(mk + (d+1)(m-1))] \\
+ c_1 [e_0 \gamma^w_{\gamma}(mk - 1) + e_1 \gamma^w_{\gamma}(mk) + \ldots + e_{(d+1)\times(m-1)} \gamma^w_{\gamma}(mk + (d+1)(m-1) - 1)] \\
\quad + \ldots \ldots \\
+ e_{(d+1)\times(m-1)} \gamma^w_{\gamma}(mk - (d+1)(m-1) + (d+1)(m-1) - 1) \\
= c_0 [(1 + F + \ldots + F^{m-1})^{d+1} \gamma^w_{\gamma}(mk)] + c_1 [(1 + F + \ldots + F^{m-1})^{d+1} \gamma^w_{\gamma}(mk - 1)] + \ldots \ldots \\
\quad + c_{(d+1)\times(m-1)} [(1 + F + \ldots + F^{m-1})^{d+1} \gamma^w_{\gamma}(mk - (d+1)(m-1))] \\
= (1 + F + \ldots + F^{m-1})^{(d+1)} [c_0 \gamma^w_{\gamma}(mk) + c_1 \gamma^w_{\gamma}(mk - 1) + \ldots + c_{(d+1)\times(m-1)} \gamma^w_{\gamma}(mk - (d+1)(m-1))] \\
= (1 + F + \ldots + F^{m-1})^{(d+1)} (1 + L + \ldots + L^{m-1})^{(d+1)} \gamma^w_{\gamma}(mk) \\
= F^{(d+1)\times(m-1)} (1 + L + \ldots + L^{m-1})^{d+1} \gamma^w_{\gamma}(mk) \\
= (1 + L + \ldots + L^{m-1})^{d+1} \gamma^w_{\gamma}(mk + (d+1)(m-1)).
\]
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Table 1: Unidirectional causality: Numerical results from analytical expressions for $p \lim \hat{\alpha}_1^*$ when $\alpha_2 = 0$, $\beta_1 = 1$ and $\rho = 1 + \alpha_1$

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These values are the same for $\beta_2 > 0.$
Table 2: Unidirectional Causality: Numerical results from analytical expressions for 
\( t(\hat{\alpha}_1^*) \) when \( \alpha_2 = 0, \beta_1 = 1 \) and \( \rho = 1 + \alpha_i \)

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These values are the same for \( \beta_2 > 0 \).
Table 3: Unidirectional Causality: Numerical results from analytical expressions for $p \lim \hat{\alpha}_2^*$ when $\alpha_2 = 0, \beta_1 = 1$ and $\rho = 1 + \alpha_i$

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Table 4: Unidirectional Causality: Numerical results from analytical expressions for $t(\hat{a}_2^*)$ when $\alpha_2 = 0$, $\beta_1 = 1$ and $\rho = 1 + \alpha_1$

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Table 5: Mutual Causality: Numerical results from analytical expressions for 
\( t(\hat{\alpha}_2^*) \) when \( \beta_1 = 1, \rho = 0 \) and \( \alpha_2 = (\rho - 1 - \alpha_1) / \beta_2 \)

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<td>-5.5</td>
<td>-4.5</td>
<td>-3.2</td>
<td>-1.9</td>
<td>-0.3</td>
<td>1.5</td>
<td>3.6</td>
<td>6.1</td>
<td>8.9</td>
<td>12.3</td>
</tr>
<tr>
<td>-2</td>
<td>-4.8</td>
<td>-3.7</td>
<td>-2.3</td>
<td>-0.8</td>
<td>0.9</td>
<td>2.9</td>
<td>5.0</td>
<td>7.5</td>
<td>10.2</td>
<td>12.9</td>
</tr>
<tr>
<td>-1</td>
<td>-3.4</td>
<td>-1.8</td>
<td>0.0</td>
<td>2.0</td>
<td>4.1</td>
<td>6.3</td>
<td>8.6</td>
<td>10.8</td>
<td>12.8</td>
<td>14.5</td>
</tr>
</tbody>
</table>

For \( \beta_2 > 0 \) the table entries are the same with the opposite sign.
Table 6. Estimated adjustment coefficients  
Cointegrating relation: \(\log(\text{Yen}/\text{DM})-\log(\text{US$}/\text{DM})+\log(\text{US$}/\text{Yen})\)

<table>
<thead>
<tr>
<th>Adjustment coefficients</th>
<th>Daily rates</th>
<th>Weekly rates End of period</th>
<th>Weekly rates Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_1)</td>
<td>-0.917*</td>
<td>-0.889 (0.236)</td>
<td>-1.940* (0.202)</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.236)</td>
<td></td>
</tr>
<tr>
<td>(\alpha_2)</td>
<td>-0.046</td>
<td>-0.116 (0.152)</td>
<td>-0.326 (0.182)</td>
</tr>
<tr>
<td></td>
<td>(0.025)</td>
<td>(0.152)</td>
<td></td>
</tr>
<tr>
<td>(\alpha_3)</td>
<td>0.050</td>
<td>-0.254 (0.258)</td>
<td>0.666 (0.301)</td>
</tr>
<tr>
<td></td>
<td>(0.037)</td>
<td>(0.258)</td>
<td></td>
</tr>
<tr>
<td>Sample size</td>
<td>922</td>
<td>184</td>
<td>184</td>
</tr>
</tbody>
</table>

Note: Numbers in parentheses are standard errors. * indicates the absolute values bigger than 2SE. If \(\alpha_2\) and \(\alpha_3\) are restricted to zero the estimates of \(\alpha_1\) in columns 3 and 4 move closer to minus unity.

Table 7. Estimates for car quota premium and stock price example

<table>
<thead>
<tr>
<th>(\hat{\alpha}_1)</th>
<th>(\hat{\alpha}_2)</th>
<th>(\hat{\beta})</th>
<th>(r_{12})</th>
<th>(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>-0.191 (0.048)</td>
<td>-0.003 (0.007)</td>
<td>-3.71</td>
<td>-0.02</td>
</tr>
<tr>
<td>m=2</td>
<td>-0.230 (0.062)</td>
<td>-0.016 (0.013)</td>
<td>-3.56</td>
<td>-0.10</td>
</tr>
<tr>
<td>m=3</td>
<td>-0.342 (0.093)</td>
<td>-0.015 (0.021)</td>
<td>-3.74</td>
<td>-0.02</td>
</tr>
<tr>
<td>m=4</td>
<td>-0.368 (0.109)</td>
<td>-0.046 (0.026)</td>
<td>-2.79</td>
<td>0.12</td>
</tr>
<tr>
<td>m=5</td>
<td>-0.483 (0.133)</td>
<td>-0.027 (0.036)</td>
<td>-3.06</td>
<td>-0.04</td>
</tr>
<tr>
<td>m=6</td>
<td>-0.572 (0.095)</td>
<td>-0.088 (0.044)</td>
<td>-3.25</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

\(r_{12}\) is the contemporaneous correlation of residuals. \(N\) is the effective sample size. The numbers in parentheses are standard errors.
Figure 1. Deviations of logarithms of daily Yen/DM direct rate from the cross rate
Figure 2. Budget surplus as a ratio of GNP (%)

Figure 3. Cointegrating relation between tax rate and spending rate
Note: Outer lines show the 2SE confidence bands. The error correction term is $z_{t-1}$.

Figure 4. OLS recursive estimates of the tax equation.
Note: Outer lines show the 2SE confidence bands. The error correction term is $z_{t-1}$.

Figure 5. OLS recursive estimates of the spending equation.