Recovering Pre ipsative Latent Factor Scores from Ipsative Measures: A Simulation Study

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A revised version of the presentation was published as:
Introduction

- Ipsative data are observed in psychological research sometimes.

- The sum of the scores over the variables for each respondent is a constant (Clemans, 1966)

\[ \sum_{i=1}^{p} x_i = 1' x = c \]
Characteristics appealing to researchers

- *Person*-centered rather than *population*-centered: consistent with the personality theories proposed by Allport (1937) and Cattell (1950)

- Effective in minimizing response bias, social desirability and faking (Baron, 1996; Cheung & Chan, 2002; Cunningham, Cunningham, & Green, 1977; Gurwitz, 1987)
Examples of ipsative measurements

- French and Raven’s (1959) Power Bases
- Kolb’s (1985) Learning Style Inventory
- Edwards’ (1959) Personal Preference Schedule
- Occupational Personality Questionnaires 4.2 Concept (Saville & Holdsworth Ltd., 1998)
Problems of ipsative data

- Ipsative data (x) are constrained:

- Population and sample covariance matrices are both singular
  \[ \Sigma_x 1 = 0 \quad \text{and} \quad 1' \Sigma_x = 0' \]
  \[ S_x 1 = 0 \quad \text{and} \quad 1' S_x = 0' \]

- Most multivariate procedures, especially factor analysis
  are incorrect (e.g., Cornwell & Dunlap, 1994; Dunlap & Cornwell, 1994; but see Ten Berge, 1999 for a legitimate case)
Research question

- Although Chan and Bentler (1993, 1996) proposed a method to analyze ipsative data in the context of CFA, ipsative scores are still not useful.

- We cannot use ipsative scores to conduct further data analyses such as regression and ANOVA
Objectives of this study

- To estimate the factor scores based on ipsative scores
- To evaluate these factor scores using simulation studies
Method to analyze ipsative data

- Chan and Bentler (1993) proposed a method to analyze ipsative data in CFA model

- Assuming that the ipsative data ($x$) is transformed from the preipsative data ($y$):

\[
x = y - 1 \bar{y} \\
= Ay, \\
\text{where } A = (I - pp^{-1}11')
\]
This type of ipsative data is called additive ipsative data.

It can be shown that

\[
\Sigma_x = A \Sigma_y A' \\
= A (\Lambda_y \Phi_y \Lambda_y' + \psi_y) A' \\
= \Lambda_x \Phi_x \Lambda_x' + \psi_x \\
= \sum_x (\Theta)
\]
An example

The preipsative model of $\mathbf{y}$ is

$$
\Lambda_y = \begin{bmatrix}
\alpha_1 & 0 \\
\alpha_2 & 0 \\
\alpha_3 & 0 \\
0 & \beta_4 \\
0 & \beta_5 \\
0 & \beta_6
\end{bmatrix}
$$

$$
\Phi_y = \begin{bmatrix}
\varphi_{11} \\
\varphi_{21} & \varphi_{22}
\end{bmatrix}
$$

$$
\Psi_y = \text{diag}[\psi_{11}, \psi_{22}, \psi_{33}, \psi_{44}, \psi_{55}, \psi_{66}]
$$
The ipsative model of $\mathbf{x}$ becomes

$$
\Lambda_x = \mathbf{A}\Lambda_y = 
\begin{bmatrix}
    \alpha_1 - \alpha & -\beta \\
    \alpha_2 - \alpha & -\beta \\
    \alpha_3 - \alpha & -\beta \\
    -\alpha & \beta_4 - \beta \\
    -\alpha & \beta_5 - \beta \\
    -\alpha & \beta_6 - \beta
\end{bmatrix}
$$

$$
\Phi_x = \Phi_y = 
\begin{bmatrix}
    \varphi_{11} \\
    \varphi_{21} \\
    \varphi_{22}
\end{bmatrix}
$$

$$
\mathbf{A}_e x = 
\begin{bmatrix}
    +.833 e_1 & -.167 e_2 & -.167 e_3 & -.167 e_4 & -.167 e_5 & -.167 e_6 \\
    -.167 e_1 & +.833 e_2 & -.167 e_3 & -.167 e_4 & -.167 e_5 & -.167 e_6 \\
    -.167 e_1 & -.167 e_2 & +.833 e_3 & -.167 e_4 & -.167 e_5 & -.167 e_6 \\
    -.167 e_1 & -.167 e_2 & -.167 e_3 & +.833 e_4 & -.167 e_5 & -.167 e_6 \\
    -.167 e_1 & -.167 e_2 & -.167 e_3 & -.167 e_4 & +.833 e_5 & -.167 e_6 \\
    -.167 e_1 & -.167 e_2 & -.167 e_3 & -.167 e_4 & -.167 e_5 & +.833 e_6
\end{bmatrix}
$$
Important characteristics

• The factor loadings are deviational factor loadings;

• The random errors are correlated systematically as the $A$ matrix; and

• The factor covariance matrix is unchanged (important!).
Chan and Bentler (1993, 1996)

Steps to analyze ipsative models:
- Propose a preipsative model $\Sigma_y(\Theta)$;
- Convert into ipsative model by applying suitable within-group constraints $\Sigma_x(\Theta)$;
- Analyze with SEM software;
- Assess model fit; and
- Convert ipsative parameter estimates and their standard errors into preipsative parameter estimates and their standard errors.
Literatures related to this approach

- Statistical theory (Chan & Bentler, 1996)
- Simulated data (Chan & Bentler, 1996; Cheung, 1997)
- Real data (Chan, 2003; Chan & Bentler, 1993; Cheung & Chan, 2002)
Recall that

- The factor covariance matrix remains unchanged after ipsative transformation:

\[
\Sigma_x = A \Sigma_y A' \\
= A (\Lambda_y \Phi_y \Lambda_y' + \Psi_y) A' \\
= \Lambda_x \Phi_x \Lambda_x' + \Psi_x \\
= \Sigma_x(\Theta)
\]

- If we can estimate the factor scores based on ipsative data, the properties of the ipsative factor scores (IFSs) should be similar to the true factor scores (TFSs)
Applications of factor scores

- Nonlinear factor analysis (McDonald, 1962)

- Interaction models for latent variables (Jöreskog, 2000; Wall & Amemiya, 2000, 2003)

- Predictive validity (Muthén & Hsu, 1993; Hsu, 1995)
Estimation of factor scores

The general formula is

\[ \hat{F} = YM \]

where \( F \) is the estimated factor scores, \( Y \) is the raw data and \( M \) is the factor score estimator.
Bartlett’s (1937) estimator

- It is a weighted least squares (WLS) estimator:

\[ \hat{F}_{WLS} = Y\Psi^{-1}\Lambda(\Lambda'\Psi^{-1}\Lambda)^{-1} \]

- Problem: the error covariance matrix should be nonsingular.
Bentler and Yuan’s (1997) estimator

- Bentler and Yuan modified the WLS estimator and proposed a generalized least squares (GLS) estimator,

\[ \hat{F}_{GLS} = Y\Sigma^{-1}\Lambda(\Lambda'\Sigma^{-1}\Lambda)^{-1} \]

- This estimator is still valid even the error matrix is singular.
After analyzing the ipsative model, we have:

\[
\Lambda_x \quad \Phi_x \quad \Psi_x \quad \hat{\Sigma}_x
\]

We can substitute these values to estimate the IFSs
Estimation of ipsative factor scores

By analyzing the ipsative data with Chan and Bentler (1993), we can estimate the IFSs by

\[
\hat{F}_{\text{WLS (ipsative)}} = X (\Psi_x)^{-1} \Lambda_x (\Lambda_x' (\Psi_x)^{-1} \Lambda_x)^{-1}
\]

\[
\hat{F}_{\text{GLS (ipsative)}} = X (\Sigma_x)^{-1} \Lambda_x (\Lambda_x' (\Sigma_x)^{-1} \Lambda_x)^{-1}
\]
A common CFA model with within-group constraints on the factor loadings and error variances while the factor covariances and factor scores are preserved.

No need to convert the ipsative parameter estimates back into preipsative parameter estimates
A simulation study

- To study the correlations between IFSs and TFSs and raw factor scores (RFSs) in different conditions

- Multivariate normal data were generally by EQS 6 (Bentler, 2001)

- Model: A two-factor CFA model was used (Chan & Bentler, 1996)
Design

- **Number of Indicators per Latent Factor (NIF):** 3, 4 and 6. The factor loadings were fixed at:
  - $NIF=3$: 1.0, 2.0 and 3.0
  - $NIF=4$: 1.0, 2.0, 3.0 and 4.0
  - $NIF=6$: 1.0, 2.0, 3.0, 1.0, 2.0 and 3.0

- Totally, there were 6, 8 and 12 variables.

- Factor variances: 1.0; Factor correlation: .5
- **Reliability of Items (RI):** 0.50, 0.70 and 0.90

- Since the factor loadings were fixed, the error variances were used to adjust for different levels of RI:

\[
\psi = \frac{\lambda^2 (1 - \rho_{xx'})}{\rho_{xx'}}
\]
Sample Size \((N)\): 200, 500 and 1,000

To summarize, the design was: \(NIF(3) \times RI(3) \times N(3)\).

500 replications were used in this study.
Assessment of the Empirical Performance

- The means and standard deviations (SDs) of the correlations between the ipsative factor scores and true factor scores ($r_{IT}$) and raw factor scores ($r_{IR}$)
## Results

### Table 1

<table>
<thead>
<tr>
<th>Reliability of items</th>
<th>Sample size</th>
<th>Ipsative factor scores</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>TFS1</td>
<td>TFS2</td>
<td>RFS1</td>
</tr>
<tr>
<td>.50</td>
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<td>IFS2</td>
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</table>

Number of indicators per latent factor = 3

**Note.** IFS: factor scores based on ipsative data; TFS: True factor scores; RFS: factor scores based on raw data. Correlations with underline represent the correlations for the corresponding factors, i.e., ipsative factor 1 for true and raw factor 1 and ipsative factor 2 for true and raw factor 2. Correlations without underline represent the correlations between factors 1 and 2 with different methods.
<table>
<thead>
<tr>
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<th>Standard deviation</th>
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<td>Standard deviation</td>
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<td>0.96534</td>
<td>0.47546</td>
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</table>

Number of indicators per latent factor = 6

<p>| .50                 | 200         | IFS1 | 0.79137 | 0.38610 | 0.85818 | 0.35778 | 0.00084 | 0.00557 | 0.00064 | 0.00644 |
|                     | IFS2        | 0.39032 | 0.78888 | 0.36260 | 0.85738 | 0.00533 | 0.00083 | 0.00617 | 0.00061 |
| 500                 | IFS1        | 0.79746 | 0.39306 | 0.86293 | 0.36569 | 0.00031 | 0.00219 | 0.00022 | 0.00231 |
|                     | IFS2        | 0.39374 | 0.79680 | 0.36447 | 0.86328 | 0.00220 | 0.00027 | 0.00243 | 0.00020 |
| 1000                | IFS1        | 0.80034 | 0.39779 | 0.86610 | 0.36857 | 0.00013 | 0.00096 | 0.00010 | 0.00102 |
|                     | IFS2        | 0.39892 | 0.80088 | 0.36958 | 0.86581 | 0.00104 | 0.00014 | 0.00118 | 0.00010 |</p>
<table>
<thead>
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<td></td>
<td></td>
</tr>
<tr>
<td>.70</td>
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<td></td>
<td>IFS2 0.48541</td>
<td>0.97051</td>
<td>0.48097</td>
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</tbody>
</table>
The estimated IFSs are positively correlated with TFSs and RFSs.

The smallest and largest $r_{IT}$ are 0.65 ($NIF=3$, $RI=.50$ and $N=200$) and 0.97 ($NIF=6$, $RI=.90$ and $N=1,000$), respectively.
When $NIF$ or $RI$ increases, $r_{IT}$ and $r_{IR}$ also increase.

When the correlations between IFSs and TFSs are smaller than 0.707, it is possible that two sets of estimated factor scores are totally uncorrelated because of the factor score indeterminacy (e.g., Guttman, 1955).
It is possible to achieve correlations higher than this value:

- When $NIF=3$, $RI > 0.70$
- When $NIF=4$ or $6$, $RI > 0.50$
\( N \) has little effect on \( r_{IT} \) and \( r_{IR} \) while it affects the \( SDs \) of \( r_{IT} \) and \( r_{IR} \).

Actually, the \( SDs \) of the correlations are very small, e.g., the largest \( SDs \) of \( r_{IT} \) and \( r_{IR} \) are only 0.00293 and 0.00497, respectively.
Discussion

- It is not easy to accept that we can recover the preipsative information (IFSs in this study) from ipsative data.

- It is suggested in this paper that we can estimate “preipsative” information from the “patterns” of ipsative scores.
An illustration

<table>
<thead>
<tr>
<th>IFS1</th>
<th>IFS2</th>
<th>Items loaded on F1</th>
<th>Items loaded on F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>Low</td>
<td>+5, +3, +1</td>
<td>-5, -3, -1</td>
</tr>
<tr>
<td>Low</td>
<td>High</td>
<td>-5, -3, -1</td>
<td>+5, +3, +1</td>
</tr>
</tbody>
</table>
However, ...

<table>
<thead>
<tr>
<th>IFS1</th>
<th>IFS2</th>
<th>Items loaded on F1</th>
<th>Items loaded on F2</th>
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<tbody>
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<td>High</td>
<td>High</td>
<td>+3, -2, -1</td>
<td>+3, -2, -1</td>
</tr>
<tr>
<td>Low</td>
<td>Low</td>
<td>+3, -2, -1</td>
<td>+3, -2, -1</td>
</tr>
</tbody>
</table>
Thus, the performance of IFSs should be poorer than the factor scores based on raw data.
Future directions

- Using these IFSs to conduct other analyses, e.g., regression and ANOVA

- Using other factor score estimators, e.g., Jöreskog, 2000; Lee & Shi, 2000; Ten Berge et al., 1999)
Potential applications to other types of ipsative data (Chan, 2003):

- partially additive ipsative data (Chan & Bentler, 1996),
- weighted additive ipsative data or within-subject standardized data (Leung & Bond, 1989),
- multiplicative ipsative data or compositional data (Aitchison, 1986) and
- ordinal ipsative data or Thurstonian ranking data (Maydeu-Olivares, 1999)
End of the presentation

Thank you very much
Your comments are highly appreciated!